249. Some Characterizations of Regular Duo Rings and Semigroups

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Some ideal-theoretic characterizations of regular duo semigroups[1] and of strongly regular rings (=regular duo rings) were given by the author [2]-[5], and by the author and F. Szász [6]-[8]. In this note we shall establish several further ideal-theoretic characterizations of these classes of associative rings and semigroups.

First we prove the following criterion.

Theorem 1. *A semigroup S is a regular duo semigroup if and only if the relation*

$$L \cap R = LRS \tag{1}$$

holds for every left ideal L and every right ideal R of S.

Proof. Let S be a semigroup with property (1) for any left ideal L and any right ideal R of S. Then (1) implies

$$R = SRS \tag{2}$$

for any right ideal R of S, i.e. every right ideal R of S is two-sided. Similarly, (1) implies

$$L = LS^2 \tag{3}$$

for each left ideal L of S, that is each left ideal L of S is two-sided. Therefore S is a duo semigroup. Next we show that S is regular. For any (two-sided) ideal I of S (1) implies

$$I = I^2S = IS^2 = SIS. \tag{4}$$

Hence we get

$$I^2 = (SIS)(SIS) = SI, \tag{5}$$

and

$$I^2 = I(IS^2) = (I^2S)S = IS \tag{6}$$

for every ideal I of S. (5) and (6) imply

$$IS = SI \tag{7}$$

for any ideal I of S. Finally (4) and (7) imply the relation

$$I = ISI \tag{8}$$

for each ideal I of S. This guarantees the regularity of S (cf. Luh [9]).

Conversely, let S be a regular duo semigroup. Then we have

$$I_1 \cap I_2 = I_1 I_2 \tag{9}$$

for any couple of (two-sided) ideals of S. (9) implies

1) We adopt the notation and terminology of [1].

(10) $$I=IS=SI$$

for any ideal I of S. (9) and (10) imply

(11) $$I_1\cap I_2=I_1I_2S$$

for any couple of two-sided ideals of S. Therefore the relation (1) is true for every left ideal I_1 and every right ideal I_2 of S.

The proof of Theorem 1 is completed.

We notice that the statement of Theorem 1 remains true with associative ring instead of semigroup. The proof is analogous to that of Theorem 1.

Theorem 2. *An associative ring S is a regular duo ring if and only if the relation* (1) *holds for each left ideal L and each right ideal R of S.*

The proof of the following criterion is quite similar to that of Theorem 1.

Theorem 3. *A semigroup S is a regular duo semigroup if and only if the relation*

(12) $$L\cap R=SLR$$

holds for every left ideal L and every right ideal R of S.

Theorem 4. *An associative ring S is a regular duo ring if and only if the relation* (12) *holds for every left ideal L and every right ideal R of S.*

It may be remarked that we have some further ideal-theoretic identities any one of which characterizes the class of regular duo semigroups.

Theorem 5. *For a semigroup S the following conditions are equivalent with each other and any one of them is a necessary and sufficient condition for S to be a regular duo semigroup*:

(A)	B_1B_2S	(G)	B_1B_2I	(M)	Q_1Q_2S
(B)	SB_1B_2	(H)	IB_1B_2	(N)	SQ_1Q_2
(C)	BQS	(I)	BQI	(O)	Q_1Q_2I
(D)	QBS	(J)	QBI	(P)	IQ_1Q_2.
(E)	SBQ	(K)	IBQ		
(F)	SQB	(L)	IQB		

Remark. B, I, Q denote bi-, two-sided, and quasi-ideal of S, respectively, and the condition (A) means that $B_1\cap B_2=B_1B_2S$ for any couple of bi-ideals of S. The condition (L) means that the relation $I\cap Q\cap B=IQB$ holds for every two-sided ideal I, every quasi-ideal Q, and for every bi-ideal B of S.

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