248. Representations of 1-homology Classes of Bounded Surfaces

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Let $F_{p,q}$ be a compact connected orientable surface of genus p with q boundary components, where q may be equal to 0. Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_p, \boldsymbol{c}_1, \cdots, \boldsymbol{c}_{q-1}$ be standard generators for the 1-dim. integral homology group $H_1(F_{p,q}; Z)$. Here the $\boldsymbol{c}_i$ correspond to consistently oriented boundary circles (one is omitted because it is homologous to the sum of the others), and the $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are standard curves on $F_{p,q}$, chosen so that $\boldsymbol{a}_i \cap \boldsymbol{a}_j = \boldsymbol{b}_i \cap \boldsymbol{b}_j = \boldsymbol{a}_i \cap \boldsymbol{b}_j = \emptyset$ if $i \neq j$ and $\boldsymbol{a}_i, \boldsymbol{b}_i$ intersect nicely at one point.

In the case $q=0$, T. Kaneko [1], [2] proved the following theorem:

Kaneko's Theorem. *A non-zero homology class* $\sum_{i=1}^p \alpha_i \boldsymbol{a}_i + \sum_{i=1}^p \beta_i \boldsymbol{b}_i$ *of* $H_1(F_{p,0}; Z)$ *is representable by a simple closed curve on* $F_{p,0}$ *if and only if the greatest common divisor* $(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_p)=1$.

In the note, we generalize this theorem for bounded case as follows:

Theorem. *A non-zero homology class* $\sum_{i=1}^p \alpha_i \boldsymbol{a}_i + \sum_{i=1}^p \beta_i \boldsymbol{b}_i + \sum_{i=1}^{q-1} \gamma_i \boldsymbol{c}_i$ *of* $H_1(F_{p,q}; Z)$ *is representable by a simple closed curve on* $F_{p,q}$ *if and only if one of the following two conditions is satisfied:*

(1) *Not all the* α_i *and* β_i *are zero and the g.c.d.* $(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_p)=1$,

(2) $\alpha_i = \beta_i = 0$ *for* $1 \leqq i \leqq p$ *and* $|\gamma_i| \leqq 1$ *for* $1 \leqq i \leqq q-1$ *and all non-zero* γ_i *have the same sign.*

Proof of necessity. Let l be an oriented simple closed curve in $\mathcal{I}F_{p,q}$.[1] Fill in each boundary component with a disk, obtaining $F_{p,0}$.

If now l is not homologous to zero on $F_{p,0}$, by Kaneko's Theorem we fall into case (1) since $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_p$ form standard generators for $H_1(F_{p,0}; Z)$.

If l is now homologous to zero on $F_{p,0}$, l bounds a surface $F_{p',1}$ on $F_{p,0}$ where $0 \leqq p' \leqq p$, and so l separates $F_{p,q}$ into two surfaces, say $F_{p',q'}$ and $F_{p-p',q+2-q'}$. On $F_{p',q'}$, l or $-l$ is homologous to the sum of $\partial F_{p',q'} - l \subset \partial F_{p,q}$, so we fall into case (2) above.

Proof of sufficiency. We shall need the following elementary lemma.

Lemma. *A homology class* $\boldsymbol{a}_1 + \gamma \boldsymbol{c}_1$ *is representable by a simple*

1) $\mathcal{I}$=interior, ∂=boundary.

closed curve in $F_{p,q}$.

Proof of Lemma. Without loss of generality, we may assume that $\gamma \geqq 1$. Let $c_{1,1}$ be an oriented simple closed curve in $\mathcal{J}F_{p,q}$ such that $c_{1,1}$ is isotopic to c_1 and $c_{1,1} \cap a_1 = \emptyset$. Then we can take an oriented disk D on $F_{p,q}$ that spans a_1 and $c_{1,1}$ with discoherent orientation. Now, the homological addition of three oriented curves $a_1 + c_{1,1} + \partial D$ is an oriented simple closed curve, say a_1', representing the homology class $a_1 + c_1$, see Fig. 1. Since $a_1 + c_1 = a_1'$ is not homologous to zero and $\sum_{i=1}^{q-1} \xi_i c_i$ for any ξ_i, we can take another standard generators for $H_1(F_{p,q}; Z)$ which contains a_1' and $c_1, \cdots, c_{q-1}$. So, applying the above argument for a_1' and c_1, we have an oriented simple closed curve a_1'' representing the homology class $a_1' + c_1 = a_1 + 2c_1$.

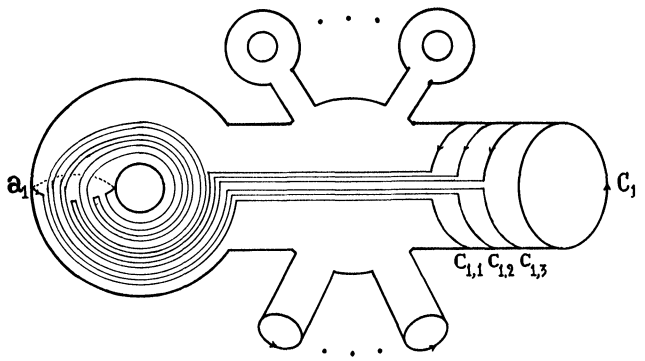

Fig. 1

Repeating of this procedure, we have an oriented simple closed curve representing the homology class $a_1 + \gamma c_1$ for any γ. (The Fig. 1 shows the case $\gamma = 3$.) Q.E.D.

We are now going to prove the sufficiency of Theorem.

(1): If $(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_p) = 1$, then the homology class $\sum_{i=1}^{p} \alpha_i a_i + \sum_{i=1}^{p} \beta_i b_i$ is representable by an oriented simple closed curve, say a_1', in $\mathcal{J}F_{p,q}$ by Kaneko's Theorem, since the arguments of [1] and [2] can be applied for a bounded surface $F_{p,q}$. Because this a_1' is not homologous to zero and to $\sum_{i=1}^{q-1} \xi_i c_i$ for any ξ_i, we can choose another standard generators for $H_1(F_{p,q}; Z)$ which contains a_1' and $c_1, \cdots, c_{q-1}$. So by Lemma, the homology class $a_1' + \gamma_1 c_1 = \sum_{i=1}^{p} \alpha_i a_i + \sum_{i=1}^{p} \beta_i b_i + \gamma_1 c_1$ is representable by a simple closed curve a_1'' in $\mathcal{J}F_{p,q}$. Of course, a_1'' is not homologous to zero and to $\sum_{i=1}^{q-1} \xi_i c_i$ for any ξ_i, and so we can choose another standard generators for $H_1(F_{p,q}; Z)$ which contains a_1'' and $c_1, \cdots, c_{q-1}$. Applying Lemma for a_1'' and c_2, we have a simple closed curve a_1''' representing the homology class $a_1'' + \gamma_2 c_2$.

Repeating of this procedure, we have a required simple closed curve on $F_{p,q}$.

(2): Suppose that α_i, β_i and γ_i satisfy the condition (2) of the Theorem. Then, by spanning the oriented boundary circles $\gamma_i \boldsymbol{c}_i$ as the same manner as the proof of Lemma, we can easily construct a required simple closed curve on $F_{p,q}$.

This completes the proof of Theorem.

References

[1] T. Kaneko, K. Aoki, and F. Kobayashi: On representations of 1-homology classes of closed surfaces. J. Fac. Sci. Niigata Univ., Ser. I, **3**, 131–137 (1963).

[2] T. Kaneko: On representations of 1-homology classes of closed surfaces. II. Sci. Rep. of Niigata Univ., Ser. A, **2**, 1–5 (1965).