198. *Continuity and Modularity of the Lattice of Closed Subspaces of a Locally Convex Space*

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1. Introduction. The set of all subspaces of a vector space forms an upper continuous modular atomistic lattice, ordered by set-inclusion. While the set $L_c(E)$ of all closed subspaces of a locally convex space E forms a complete DAC-lattice which is, in general, neither upper continuous nor modular (cf. [3], Chapter VII). The main purpose of this paper is to find some conditions on E under each of which $L_c(E)$ becomes upper continuous, lower continuous and modular respectively. Our main results are as follows: (1) $L_c(E)$ is upper continuous if and only if every subspace of E is closed, (2) $L_c(E)$ is lower continuous if and only if E is a minimal space, (3) in case E is metrisable, $L_c(E)$ is modular if and only if E is a minimal space. The last result is a generalization of a theorem in Mackey [2].

2. Continuity and modularity in DAC-lattices. A lattice L is called *upper continuous* when $a_\delta \uparrow a$ implies $a_\delta \wedge b \uparrow a \wedge b$ and called *lower continuous* when $a_\delta \downarrow a$ implies $a_\delta \vee b \downarrow a \vee b$ ([3], Definition 2.14). We write $(a, b)M$ (resp. $(a, b)M^*$) when the pair (a, b) is modular (resp. dual-modular)([3], Definition 1.1).

Lemma 1. *Let a be an element of a complete lattice L. If the interval $L[a, 1]=\{x \in L; a \leqq x \leqq 1\}$ is upper continuous then for any $b \in L$ there exists a maximal element b_1 such that $b_1 \leqq b$ and $(b_1, a)M^*$.*

An atomistic lattice L with the covering property is called an *AC-lattice* ([3], Definition 8.7). A lattice L with 0 and 1 is called a *DAC-lattice* when both L and its dual are AC-lattices ([3], Definition 27.1). In a DAC-lattice, $(a, b)M$ and $(b, a)M$ are equivalent and so are $(a, b)M^*$ and $(b, a)M^*$ ([3], Theorem 27.6).

Theorem 1. *Let L be a complete DAC-lattice and let $a \in L$. If either $L[a, 1]$ is upper continuous or $L[0, a]$ is lower continuous, then $(a, x)M$ and $(a, x)M^*$ hold for every $x \in L$.*

Corollary. *If a complete DAC-lattice L is either upper or lower continuous then L is modular.*

Lemma 2. *Let L be a complete AC-lattice and assume that there exists a sequence of atoms p_n such that $1=\vee(p_n; 1 \leqq n < \infty)$. If L is sequentially upper continuous ($a_n \uparrow a$ implies $a_n \wedge b \uparrow a \wedge b$) then it is upper continuous.*

3. Continuity of the lattice of closed subspaces of a locally convex space. Let E be a locally convex space. (Here locally convex spaces are always Hausdorff.) The set $L_c(E)$ of all closed subspaces of E forms an irreducible complete DAC-lattice, ordered by set-inclusion ([3], Theorem 31.10). We denote by E' the vector space of all continuous linear forms on E and by E^* the space of all linear forms.

Theorem 2. *Let E be a locally convex space. The following statements are equivalent.*

(α) *$L_c(E)$ is upper continuous.*

(β) *Every subspace of E is closed.*

(γ) *$E'=E^*$.*

(δ) *The topology of E is finer than or equal to $\sigma(E, E^*)$.*

Theorem 3. *Let E be a locally convex space. The following statements are equivalent.*

(α) *$L_c(E)$ is lower continuous.*

(β) *$E=E'^*$.*

(γ) *There exists no strictly coarser locally convex topology on E.*
In this case, E is called a *minimal space* ([5], p. 191).

(δ) *E is isomorphic with the product space K^I of one-dimensional spaces.*

Remark. By [1], Chap. IV, § 1, Exercise 13 c), if a subspace of a locally convex space E is minimal then it is closed in E. Moreover it can be shown (see Exercise 13 f)) that
(1) if a subspace A of E is minimal then the linear sum $A+X$ is closed for every closed subspace X of E, and that
(2) if subspaces A and B are minimal then so is $A+B$.
By [3], Theorem 31.10, $A+X$ is closed if and only if $(A, X)M^*$ in $L_c(E)$, and by Theorem 3, A is minimal if and only if the interval $\{X \in L_c(E); X \leqq A\}$ of $L_c(E)$ is lower continuous. Therefore, Theorem 1 in § 2 is a generalization of the statement (1). On the other hand, it follows from (2) that in the lattice $L_c(E)$ the set of all minimal subspaces forms a greatest lower continuous ideal.

4. Modularity of the lattice of closed subspaces. It follows from Theorems 2 and 3 and Corollary of Theorem 1 that $L_c(E)$ is modular if either $E'=E^*$ or $E=E'^*$. When E is infinite-dimensional and metrisable, it does not occur that $E'=E^*$ ([5], p. 148), and moreover we shall show that $E=E'^*$ is the only case that $L_c(E)$ is modular.

Lemma 3. *Let E be a metrisable locally convex space. If for some sequence of elements $A_n \in L_c(E)$ such that $A_n \downarrow \{0\}$ there exists a neighborhood U of 0 such that $A_n \not\subset U$ for every n, then $L_c(E)$ is not modular.*

Lemma 4. *Let E be a metrisable locally convex space. If $L_c(E)$ is modular then it is sequentially lower continuous.*

Lemma 5. *Let E be a separable metrisable locally convex space. If $L_c(E)$ is sequentially lower continuous then it is lower continuous.*

Lemma 6. *Let E be a locally convex space such that every separable closed subspace of E is metrisable. If $L_c(E)$ is sequentially lower continuous then the completion $\hat{E}$ of E is minimal.*

Theorem 4. *Let E be a metrisable locally convex space. The following statements are equivalent.*

(α) *$L_c(E)$ is modular.*

(β) *$L_c(E)$ is lower continuous.*

(γ) *E is minimal* (E is isomorphic to K^I where I is countable).

This theorem is a generalization of Mackey [2], Theorem III–15. Moreover, this theorem implies the following result which is a generalization of [2], Theorem III–14 and is closely related to Martineau's result in [4]. *If E is a metrisable locally convex space which is not minimal, then there exist two closed subspaces A and B of E such that $A+B$ is not closed.*

In this paper, the proofs are omitted. The details will be stated in another paper.

References

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