186. *A Semigroup-Theoretic View of Projective Class Groups*

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This note has arisen from an interest in subsumming the basic description of the projective class groups of rings and their representation as quotients of Grothendieck groups into the elementary theory of commutative semigroups. This goal is reached in § 2 by means of a method given in § 1 of constructing quotient monoids modulo subtractive subsemigroups. The quotient construction of § 1 provides a convenient vocabulary for a discussion of greatest monoid images which is given in § 3. An interpretation of § 1 and § 3 in terms of category concepts is appended as § 4. All semigroups considered here will be commutative and additive notation will be used.

1. The quotient monoids. A subsemigroup B of a commutative semigroup A is *subtractive* if whenever $a+b=b'$, for $a \in A$ and b, $b' \in B$, we have $a \in B$. To give familiarity with the definition we list three elementary observations: The subtractive subsemigroups of a group are precisely the subgroups. A proper ideal is never subtractive. If C is a subtractive subsemigroup of B and B is a subtractive subsemigroup of A, then C is subtractive in A.

Let B be a subtractive subsemigroup of a commutative semigroup A. We use B to define a relation $\rho(B)$ in A: For each a, $a' \in A$ we write $a\rho(B)a'$ if there are b, $b' \in B$ for which $a+b=a'+b'$. It is easy to verify that $\rho(B)$ is a congruence relation and that whenever we have $a\rho(B)b$, for $a \in A$ and $b \in B$, we have $a \in B$. We denote the quotient semigroup $A/\rho(B)$ by the shorter form A/B and we observe that A/B is a monoid with B as identity. If A is a group, then A/B coincides with its usual meaning in group theory.

For each homomorphism $h: A \to M$ of a commutative semigroup A onto a monoid M, we define the kernel of h to be the subset $\ker(h) = \{a \in A \mid h(a)=0\}$. We list three elementary observations concerning kernels: Each kernel is a subtractive subsemigroup. For each subtractive subsemigroup B of each commutative semigroup A, the kernel of the natural map $A \to A/B$ is B. A subsemigroup B of a commutative semigroup A is the kernel of a homomorphism of A onto a monoid if

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and only if B is subtractive.

Our fundamental tool is the following universal property of natural maps onto quotients.

Lemma 1. *If $h: A \to M$ is a homomorphism of a commutative semigroup A onto a monoid M and B is a subtractive subsemigroup of $\ker(h)$, then $h = h'k$ where k is the natural map $A \to A/B$ and h' is a homomorphism $A/B \to M$.*

Proof. Let $a\rho(B)a'$. Then there are $b, b' \in B$ for which $a+b = a'+b'$ and we compute: $h(a) = h(a) + h(b) = h(a+b) = h(a'+b') = h(a') + h(b') = h(a')$. Thus h induces a function $h': A/B \to M$ for which $h = h'k$ and such a function is necessarily a homomorphism.

Our next objective is to describe the smallest subtractive subsemigroup containing a given subsemigroup. Accordingly, for subsemigroups X and Y of a commutative semigroup A, we define: $X - Y = \{a \in A \mid a + y = x \text{ for some } x \in X \text{ and } y \in Y\}$. The following three observations are easily verified: $X - Y$ is a subsemigroup. $X - X$ is subtractive. Each subtractive subsemigroup of A that contains X also contains $X - X$. These observations together with Lemma 1 give: *For each subsemigroup B of A, every homomorphism h of A onto a monoid which satisfies $h(B) = 0$ possesses the natural map $A \to A/(B-B)$ as a right factor.*

Example. Let R be a ring. Let A be the set of isomorphism types of finitely generated left R-modules. The operation of forming the direct sum of two modules provides an operation for A with respect to which A is a commutative monoid. Let B be the cyclic subsemigroup of A generated by the isomorphism type of R. We make four interpretations: B consists of the isomorphism types of the finitely generated free modules. $B - B$ consists of the types of those summands of finitely generated free modules which have free complementary summands. $B - A$ consists of the types of finitely generated projective modules. Finally, the quotient monoid $(B-A)/(B-B)$ is a group and it may be recognized as the (left) *projective class group of the ring R* [2, § 6].

We are interested in viewing the projective class groups of rings in as broad a context as possible within the theory of commutative semigroups. The quotient $(B-A)/(B-B)$ is easily recognized as the group of units of the monoid $A/(B-B)$. Our attention will therefore be focused on the group of units of an arbitrary quotient monoid. The projective class groups are commonly discussed in terms of Grothendick groups and our discussion of groups of units will extend the known relationship for the projective class groups.

2. The group of units of a quotient monoid. Throughout this paragraph A will denote a commutative semigroup and B a subsemigroup of A. Our object of study is the group of units of the quotient monoid $A/(B-B)$. There is no difficulty in locating the units: The group of units of $A/(B-B)$ is $(B-A)/(B-B)$. Our interest centers on relating the structure of this group to Grothendieck groups.

We will use a specific description of the Grothendieck groups: Let X be a commutative semigroup. Let σ be the relation in X defined by: $x\sigma x'$ if $x+x''=x'+x''$ for some $x'' \in X$. Then σ is a congruence relation and is the finest one for which X/σ is cancellative. By the Grothendieck group of X we will mean the group, Gro (X), of fractions of X/σ. The natural map $X\rightarrow X/\sigma$ followed by the inclusion $X/\sigma\subset \text{Gro}(X)$ gives us a cannonical homomorphism $g: X\rightarrow \text{Gro}(X)$. Each homomorphism $X\rightarrow Y$ between commutative semigroups induces a homomorphism $X/\sigma\rightarrow Y/\sigma$ which extends uniquely to a homomorphism Gro $(X)\rightarrow$ Gro (Y). A commutative diagram results:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X/\sigma & \longrightarrow & Y/\sigma \\ \downarrow & & \downarrow \\ \text{Gro}(X) & \longrightarrow & \text{Gro}(Y). \end{array}$$

One consequence is the following standard characterization of the Grothendieck groups: *Any homomorphism of X into a group possesses the map $g: X\rightarrow \text{Gro}(X)$ as a right factor.*

Theorem. *For each subsemigroup B of each commutative semigroup A the group $(B-A)/(B-B)$ of units of $A/(B-B)$ is isomorphic with* Gro $(B-A)/$Gro (B).

We begin the proof by explaining the manner in which we may regard Gro (B) as a subgroup of Gro $(B-A)$. The inclusions $B\subset B-B \subset B-A$ provide homomorphisms Gro $(B)\rightarrow$Gro $(B-B)\rightarrow$Gro $(B-A)$. That these two maps are monomorphisms is a consequence of:

Lemma 2. *If X and Y are subsemigroups of A for which $X\subset Y \subset X-A$ then the map* Gro $(X)\rightarrow$Gro (Y) *provided by $X\subset Y$ is a monomorphism.*

Proof. It is enough to verify that the related map $X/\sigma\rightarrow Y/\sigma$ is one-one. To do this we suppose that $x+y=x'+y$ for some $x, x' \in X$ and $y\in Y$. Since $Y\subset X-A$ there is an $x''\in X$ and an $a\in A$ for which $y+a=x''$. Then $x+x''=x'+x''$ from which the one-one property follows.

One next concern is to verify that the map Gro $(B)\rightarrow$Gro $(B-B)$ is an isomorphism. For this purpose we will refer to the commutative diagram:

$$\begin{array}{ccc} B & \subset & B-B \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ \mathrm{Gro}\,(B) & \xrightarrow{h} & \mathrm{Gro}\,(B-B). \end{array}$$

Each element of $\mathrm{Gro}\,(B-B)$ is of the form $g_2(a_1)-g_2(a_2)$ with a_1, $a_2 \in B-B$. There are $b_1, b_1', b_2, b_2' \in B$ such that $a_1+b_1=b_1'$ and $a_2+b_2=b_2'$. We compute: $g_2(a_1)-g_2(a_2)=g_2(b_1')-g_2(b_1)-g_2(b_2')+g_2(b_2)=hg_1(b_1')-hg_1(b_1)-hg_1(b_2')+hg_1(b_2)$, which is in the image of h.

On replacing the monomorphisms with inclusion maps we have the commutative diagram:

$$\begin{array}{ccccccccc} B & \subset & B-B & \subset & B-A & \longrightarrow & (B-A)/(B-B) & & \\ \downarrow & & \downarrow & & \downarrow & {\scriptstyle j}\nearrow & \downarrow{\scriptstyle i} & \uparrow{\scriptstyle k} & \\ \mathrm{Gro}\,(B) & = & \mathrm{Gro}\,(B-B) & \subset & \mathrm{Gro}\,(B-A) & \to & \mathrm{Gro}\,(B-A)/\mathrm{Gro}\,(B), & & \end{array}$$

where i, j, and k are constructed as follows: The composite map $B-A \to \mathrm{Gro}\,(B-A)/\mathrm{Gro}\,(B)$ annihilates $B-B$ and therefore also has a monoid as its image. Consequently by Lemma 1 a homomorphism i is induced. Since $(B-A)/(B-B)$ is a group, a homomorphism j is induced. Since j must annihilate the image of $B-B \to \mathrm{Gro}\,(B-B)$ and since this image generates $\mathrm{Gro}\,(B-B)$, it follows that j annihilates $\mathrm{Gro}\,(B-B)=\mathrm{Gro}\,(B)$. Consequently, a homomorphism k is induced. A chase of the diagram will confirm that both ik and ki are identity maps. The chase is facilitated by observing that the image of $B-A$ in $\mathrm{Gro}\,(B-A)/\mathrm{Gro}\,(B)$ is a generating set. This completes the proof of the theorem.

Assume now that B is cyclic, which is the case in the example concerning the projective class groups. If B is infinite, then $\mathrm{Gro}\,(B) \cong Z$, the additive group of integers. In this case the group of units of $A/(B-B)$ has the form $\mathrm{Gro}\,(B-A)/Z$. If B is finite with b as a generator, then there exist positive integers m and n which are least subject to $mb=(m+n)b$. In this case $\mathrm{Gro}\,(B) \cong Z_n$, the additive group of integers modulo n, and the group of units has the form $\mathrm{Gro}\,(B-A)/Z_n$.

3. Greatest monoid images. A homomorphism $h: A \to M$ of a commutative semigroup A onto a monoid M is a *greatest homomorphism of A onto a monoid* if every homomorphism of A onto a monoid possesses h as a right factor. When h is such a greatest homomorphism, we call *M a greatest monoid image of A*. The general study of greatest images of given type has been initiated in [1, § 11.6] and [3]. The quotient monoid construction provides a vocabulary that allows an elementary discussion of the existence and structure of greatest monoid images of commutative semigroups. The discussion will be given in terms of the following concept: An element p of a commutative semigroup A will be called a *pre-identity* if for every $a \in A$ there are positive integers m and n for which $p+ma=na$.

Proposition. *Let A be a commutative semigroup and P be the set of pre-identities of A. If P is not empty then P is a subtractive subsemigroup of A and A/P is a greatest monoid image of A. If P is empty then A has no greatest monoid image.*

Proof. The elementary verifications of $P \pm P \subset P$ will be omitted. Suppose that P is not empty and that $h: A \to M$ is a homomorphism of A onto a monoid. Let p be any element of P and let a be an element of A for which $h(a)=0$. For some positive integers m, n we have $p+ma=na$ which yields $h(p)=0$. Since $P \subset \ker(h)$, we conclude by Lemma 1 that the natural map $A \to A/P$ is a right factor of h as required.

Suppose now that $h: A \to M$ is a greatest map of A onto a monoid and that p is an element of $\ker(h)$. To complete the proof we need only show that p is a pre-identity. Let a be any element of A and let B be the cyclic subsemigroup it generates. The natural map $A \to A/(B-B)$ must possess h as a right factor and consequently we have $p \in B-B$. Thus $p+ma=na$ for some positive integers m, n and we conclude that p is a pre-identity.

4. A categorical footnote. Let $\mathcal{A}$ be the category whose objects are commutative semigroups and whose morphisms consist of the identity maps and of those homomorphisms $h: X \to Y$ for which Y is a monoid and $0 \in h(X)$. Thus $\mathcal{A}$ contains the category $\mathcal{C}$ of commutative monoids as a full subcategory. The quotient monoids of § 1 are essentially the cokernels of $\mathcal{A}$: If $h: X \to Y$ is a morphism in $\mathcal{A}$ and $B=h(X)$, then we observe from Lemma 1 that the natural map $Y \to Y/(B-B)$ is a cokernel for h. When Y is a monoid and $A=\ker(h)$, Lemma 1 also provides a canonical factorization expressed by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow & & \cup \\ X/A & \xrightarrow{h'} & B \end{array}$$

where h' is a subjective homomorphism with trivial kernel. We may interpret § 3 as a determination of those objects in $\mathcal{A}$ which possess reflections in $\mathcal{C}$. Accordingly, § 3 shows that the full subcategory $\mathcal{B}$ of $\mathcal{A}$, having as its objects the commutative semigroups with pre-identities, is the unique maximal subcategory of $\mathcal{A}$ which coniains $\mathcal{C}$ as a reflective subcategory of itself.

We close with a question: Which commutative semigroups S have the property that for each commutative semigroup A and each subtractive subsemigroup $B \subset A$, the homomorphism $S \otimes B \to S \otimes A$ induced by the inclusion is one-one?

References

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