225. Results Related to Closed Images of M-Spaces. II

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This is a continuation of our paper [4]. Supplements to the results in [4] will be given, as well as a new proof and slight extension of a theorem of Nagata [5]. We use the same notation and terminology as in our previous paper, in particular, all spaces are assumed to be T_1-spaces.

4. Supplements and consequences of Theorem 3.1.

Proposition 4.1. *Let Y be a regular space such that conditions (a) and (b) of Theorem 3.1 are satisfied. Then the M-space X for which a closed continuous map $f: X \to Y$ exists may be chosen to be normal or paracompact according as Y is normal or paracompact.*

Proof. Let X be the space constructed in the proof of [4, Theorem 3.1]. First note that X is closed in $B \times Y$. To show this, assume $(\alpha, y) \notin X$. Then $y \notin \bigcap_{i=1}^{\infty} F_{i\alpha_i}$, with $\alpha = (\alpha_1, \alpha_2, \cdots)$. Hence there exists an $m \in N$ such that

$$y \cap \left[\bigcap_{i=1}^{m} F_{i\alpha_i}\right] = \emptyset.$$

If we take $V(y)$ as $Y - \bigcap_{i=1}^{m} F_{i\alpha_i}$, we have

$$[B(\alpha_1, \cdots, \alpha_m) \times V(y)] \cap X = \emptyset,$$

so X is closed in $B \times Y$.

Now Y is the closed image of an M-space X. Since X is a P-space by Morita [2, Theorem 6.3], so is Y by [2, Theorem 3.3]. Hence if Y is normal, $B \times Y$ is normal [2, Theorem 4.1]. Thus X is normal. On the other hand, if Y is paracompact, so is $B \times Y$. Then X is paracompact.

Remark. Note that if $\bigcap_{i=1}^{\infty} F_{i\alpha_i}$ is compact for any q-sequence $\{F_{i\alpha_i}\}$, then the map $\varphi: X \to B$ is perfect and hence X is paracompact.

Definition 4.2. A space X is said to be an *M^*-space* if and only if X has a sequence $\{\mathcal{F}_n : n \in N\}$ of locally finite closed covers satisfying condition (1) below:

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(1) every point-sequence $\{x_n\}$, where $x_n \in \mathrm{St}(x, \mathcal{F}_n)$ for every $n \in N$ and for fixed $x \in X$, has a cluster point.

Note that condition (1) of Definition 4.2 is equivalent with

(1′) if $y \in F_{i\alpha_i} \in \mathcal{F}_i$ for every $i \in N$, then the family $\{F_{i\alpha_i} : i \in N\}$ forms a q-sequence at y.

Proposition 4.3. *Let Y be a space satisfying conditions* (a) *and* (b) *of Theorem* 3.1. *Assume that for a point y of Y any sequence $\{F_{n\alpha_n}\}$, with $y \in F_{n\alpha_n} \in \mathcal{F}_n$ for all $n \in N$, is a q-sequence at y. Then* $\mathrm{Bd}\, f^{-1}(y) \subset X$ *is countably compact, where f is the map constructed in the proof of* [4, *Theorem* 3.1]. *In particular, if* $\mathrm{Bd}\, f^{-1}(y)$ *is countably compact for any point y of Y, then Y is an M^*-space.*

Proof. Assume that $\mathrm{Bd}\, f^{-1}(y)$ is not countably compact in X. Then a countable set of points $\alpha^{(i)}$, $i \in N$, exists such that $(\alpha^{(i)}, y) \in \mathrm{Bd}\, f^{-1}(y)$ and $\{\alpha^{(i)}\}$ has no cluster point in B. Then choose in B a discrete collection of open sets of the form

$$B(\alpha_1^{(i)}, \cdots, \alpha_{n_i}^{(i)}) \ni \alpha^{(i)}.$$

Assume, without loss of generality, that $n_i < n_{i+1}$ for all $i \in N$.

Since $(\alpha^{(i)}, y) \in \mathrm{Bd}\, f^{-1}(y)$, we have

$$[B(\alpha_1^{(i)}, \cdots, \alpha_{n_i}^{(i)}) \times V(y)] \cap [X - f^{-1}(y)] \neq \emptyset$$

for any neighborhood $V(y)$ in Y and for every $i \in N$. So choose a point

$$(\beta^{(1)}, y_1) \in [B(\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}) \times V(y)] - f^{-1}(y).$$

Note that, since Y is T_1, there exists a neighborhood $W_1(y)$ of y such that

$$y_1 \notin W_1(y), \qquad W_1(y) \subset V(y).$$

Now take

$$(\beta^{(2)}, y_2) \in [B(\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}) \times W_1(y)] - f^{-1}(y);$$

then a neighborhood $W_2(y)$ exists satisfying

$$y_2 \notin W_2(y), \qquad W_2(y) \subset W_1(y).$$

Continuing in this manner, we find a sequence of points $\{(\beta^{(k)}, y_k)\}$ such that

$$(\beta^{(k)}, y_k) \in [B(\alpha_1^{(k)}, \cdots, \alpha_{n_k}^{(k)}) \times W_{k-1}(y)] - f^{-1}(y),$$

where a neighborhood $W_k(y)$ exists such that

$$y_k \notin W_k(y), \qquad W_k(y) \subset W_{k-1}(y).$$

The sequence $\{(\beta^{(n)}, y_n)\}$ does not cluster, since the family $\{B(\alpha_1^{(i)}, \cdots, \alpha_{n_i}^{(i)}) : i \in N\}$ is discrete. Also, $\{(\beta^{(k)}, y_k)\}$ is closed in X. So $\{f(\beta^{(k)}, y_k)\} = \{y_k\}$ is closed in Y.

But $y_k \in F_{k\alpha_k}$ for every $k \in N$ where we put $\alpha_k = \alpha_k^{(k)}$; hence this contradicts the fact that $\{F_{k\alpha_k}\}$ forms a q-sequence.

The last statement of this theorem follows directly from [1, Theorem 2.3] and a technique given in [3].

Definition 4.4. A sequence $\{A_n\}$ of subsets of a space Y is called *a network at a point* y of Y if and only if

(1) for any neighborhood U of y there is some A_n with $y \in A_n \subset U$;

(2) $y \in A_n$ for all $n \in N$.

Then as a direct consequence of the proof of [4, Theorem 3.1] we obtain the following theorem.

Theorem 4.5. *A space Y is the closed image of a metric space if and only if the following conditions are satisfied*:

(*a*) *Y is a quasi-k-space*;

(*b*) *there exists a sequence $\{\mathcal{F}_n\}$ of hereditarily closure preserving closed covers of Y with the properties below*:

(i) *for any point y of Y, any sequence $\{A_n\}$ of sets, such that $A_n \in \mathcal{F}_n$ and $y \in A_n$ for all $n \in N$, is either hereditarily closure preserving or forms a network at y,*

(ii) *for any point y of Y there is a network $\{A_n\}$ at y such that $A_n \in \mathcal{F}_n$ for $n \in N$.*

It is obvious that Lašnev's characterization of closed images of metric spaces follows immediately from our Theorem 4.5.

5. Related results. In this section we will use the technique of proof given in [4, Theorem 3.1] to discuss some properties of spaces related to M-spaces.

Definition 5.1. A space X is said to be an *$M^{\#}$-space* if and only if X has a sequence $\{\mathcal{F}_n : n \in N\}$ of closure preserving closed covers satisfying condition (1) of Definition 4.2.

M^*-spaces were introduced by Ishii [1]; $M^{\#}$-spaces are due to Siwiec and Nagata [6]. It is well known that every M-space is M^*, and M^*-space are $M^{\#}$.

Proposition 5.2. *A space Y is M^* if and only if there exist an M-space X and a perfect map $f: X \to Y$. Furthermore, if this is the case, X can be chosen to be regular, normal or paracompact according as Y is regular, normal or paracompact.*

Proof. Let Y be an M^*-space with $\{\mathcal{F}_i\}$ the family of locally finite closed covers of Y satisfying condition (1) of Definition 4.2. Define, exactly as in the proof of Theorem 3.1, the metric space B and the M-space $X \subset B \times Y$. If the map $f: X \to Y$ is defined by $f(\alpha, y) = y$, we must first show that f is closed.

Take a closed set $A \subset X$, and let $y_0 \in \mathrm{Cl}\, f(A)$. Now

$$A = \cup \{A \cap B(\lambda) \times Y : \lambda \in \Omega_1\}.$$

Since $y_0 \in \mathrm{Cl}\, f(A) = \cup \{\mathrm{Cl}\,[f(A) \cap B(\lambda) \times Y] : \lambda \in \Omega_1\}$, an index $\alpha_1 \in \Omega_1$ exists such that

$$y_0 \in \mathrm{Cl}\,[f(A) \cap B(\alpha_1) \times Y].$$

Repeating this argument, we find indices $\alpha_1, \alpha_2, \cdots$ such that for every $n \in N$,

$$y_0 \in \mathrm{Cl}\,[f(A \cap B(\alpha_1, \cdots, \alpha_n) \times Y)] \subset \bigcap_{i=1}^{n} F_{i\alpha_i}.$$

So $\{F_{i\alpha_i}\}$ forms a q-sequence at y_0 by statement (1′). Hence $(\alpha_1, \alpha_2, \cdots) = \alpha \in B$, and $(\alpha, y_0) \in A$ exactly as in [4, Theorem 3.1]. Then $f(\alpha, y_0) = y_0$, $y_0 \in f(A)$. So f is closed.

It remains to show that $\{f^{-1}(y)\}$ is compact for every $y \in Y$. But $f^{-1}(y) = \{\beta = (\beta_1, \beta_2, \cdots) \in B : \{F_{i\beta_i}\}$ is a q-sequence at $y\} \times \{y\}$. Further, for each $i \in N$, $y \in F_{i\alpha_i}$ for at most finitely many elements of $\mathcal{F}_i$. Thus $f^{-1}(y)$ is compact, which implies that f is a perfect map.

The proof of the final assertion of this proposition is exactly the same as that of 4.1, and will be omitted.

The first part of 5.2 was originally proved by Nagata [5] using multivalued mappings; our result is a slight refinement of his.

Taking 4.3 into account, we have the following statement, whose proof is nearly the same as that of 4.3.

Proposition 5.3. *A space Y has a sequence $\{\mathcal{F}_n : n \in N\}$ of hereditarily closure preserving closed covers satisfying condition* (1) *of Definition* 4.2 *if and only if there exist an M-space X and a perfect map $f : X \to Y$.*

This raises the following question: Is every $M^{\#}$-space an M^*-space?

References

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