217. *Markov Semigroups with Simplest Interaction. I*

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In this paper we will give a formulation of semigroups with simplest interaction (briefly, interaction semigroups) in an analogous way to that of branching semigroups. The most important examples and models of this class of semigroups are the solutions of spatially homogeneous Boltzmann equations with finite cross-sections, which will be proved in Part II to be the reversed processes of some branching processes. The key idea to reduce such a nonlinear evolution equation to a linear one is based on the fact that the former equation is the equation on the maximal ideal spaces of a certain commutative Banach algebras. Another example is Burgers' equation which can be solved along this idea.

§0. Notations.

0.0. Let Q be a compact Hausdorff space with a countable basis, Q^n the n-fold direct product of the space, Q_n the n-fold symmetric direct product for each positive integer n, $Q^*=\bigcup_{n\geqq 1} Q^n$ and $Q_*=\bigcup_{n\geqq 1} Q_n$. All of these spaces are compact except Q^* and Q_* which are locally compact. For any topological space X, $\mathcal{C}(X)$ denotes the totality of continuous functions on X with usual topology and $\mathcal{M}(X)$ the totality of Borel measures on X with finite mass for each compact subset in X.

0.1. For each $n\geqq 1$, R_n is the restriction map from $\mathcal{C}(Q^*)$ onto $\mathcal{C}(Q^n)$ and I_n is the operator from $\mathcal{C}(Q^n)$ into $\mathcal{C}(Q^*)$ defined by

$$I_n f=\begin{cases} f & \text{on} \quad Q^n \\ 0 & \text{off} \quad Q^n \end{cases} \qquad \text{for} \quad f\in \mathcal{C}(Q^n).$$

The image of I_n is in $\mathcal{C}_0(Q^*)$ the totallity of elements is $\mathcal{C}(Q^*)$ which tend to zero at infinity and $\sum I_n R_n$ is the identity on $\mathcal{C}(Q^*)$ and $\mathcal{C}_0(Q^*)$. S_n is the symmetrizing operator on $\mathcal{C}(Q^n)$ i.e.

$$S_n f(x_1,\cdots,x_n)=\frac{1}{n\,!}\sum_{\sigma} f(x_{\sigma(1)},\cdots,x_{\sigma(n)})$$
$$(f\in \mathcal{C}(Q^n),\ (x_1,\cdots,x_n)\in Q^n)$$

where the summation is taken over all the permutations of $\{1,\cdots,n\}$. The image of $\mathcal{C}(Q^n)$ by S_n may naturally be identified with the space $\mathcal{C}(Q_n)$, and the images of $\mathcal{C}(Q^*)$ and $\mathcal{C}_0(Q^*)$ by $S=\sum_{n\geqq 1} I_n S_n R_n$ with the spaces $\mathcal{C}(Q_*)$ and $\mathcal{C}_0(Q_*)$, respectively.

The following operator M from $\mathcal{C}(Q)$ into $\mathcal{C}(Q^*)$ is an essential one.

$$Mf(x_1, \cdots, x_n)=f(x_1)\cdots f(x_n) \qquad \text{on } Q_n\ (n\geqq 1).$$

The image by M of the open unit ball in $\mathcal{C}(Q)$ is contained in $\mathcal{C}_0(Q^*_*)$.

0.2. The operator I^*_n, R^*_n, S^*_n, S^* and M^* are defined similarly for $\mathcal{M}(Q^n)$, $\mathcal{M}(Q^*)$, etc. S^*_n and S^* are the transposed operator of S_n and S respectively, but the others are not.

0.3. The space $\mathcal{C}(Q_*)$ is endowed with the structure of commutative algebra without unit by the multiplication $*$ defined as follows: For ϕ and ψ in $\mathcal{C}(Q_*)$,

$$\phi*\psi=\sum_n I_n S_n\Big[\sum_{i+j=n}(R_i\phi)\otimes(R_j\phi)\Big]$$

where $\otimes$ denotes the tensor product. The space $\mathcal{C}_0(Q_*)$ fails to be an algebra. An interesting subspace of $\mathcal{C}(Q_*)$ is the space

$$\Lambda=\Big\{\phi\in C(Q_*): \sum_{n\geqq 1}\|R_n\phi\|_{\mathcal{C}(Q_n)}<\infty\Big\}$$

and it becomes a commutative Banach algebra with unit whose maximal ideal space can be identified with the closed unit ball of the dual Banach space of $\mathcal{C}_0(Q_*)$, though it is not used in Part I and Part II.

0.4. The space $\mathcal{M}(Q_*)$ is also endowed with the algebra structure by the multiplication $*$ defined for μ and ν in $\mathcal{M}(Q_*)$ by

$$\mu*\nu=\sum_n I^*_n S^*_n\Big[\sum_{i+j=n}(R^*_i\mu)\otimes(R^*_j\nu)\Big].$$

§1. Definition of interaction semigroup.

1.1. Let $\mathcal{X}$ be one of the spaces $\mathcal{C}(Q_*)$, $\mathcal{C}_0(Q_*)$, Λ or the Banach space $\mathcal{C}_0(Q_*)$ of all bounded continuous functions on Q_* vanishing at infinity.

Definition. A semigroup $(\bar{T}_t)_{t\geqq 0}$ of linear operators on $\mathcal{X}$ is called an *interaction semigroup* if it satisfies the *interaction property*: if ϕ and ψ are in $\mathcal{X}$ and if $\phi*\psi$ belongs to $\mathcal{X}$, then,

(1) $$\bar{T}_t(\phi*\psi)=\bar{T}_t\phi*\bar{T}_t\psi \qquad \text{for any } t\geqq 0.$$

1.2. Remark. a) This property (1) is in a close relation to the branching property:

(2) $$\bar{T}_t Mf=MR_1\bar{T}_1 Mf \qquad \text{for any } t\geqq 0$$

if f is an element of $\mathcal{C}(Q)$ such that Mf is in $\mathcal{X}$. (See Theorem 2.1.)

b) It is easily proved that (1) is equivalent to the relation

(1′) $$\bar{T}_t\bar{M}=\bar{T}\bar{M}_t$$

on $\{\phi\in X: \bar{M}\phi\in X\}$ where $\bar{M}\phi=\sum_{n\geqq 1}\phi^n$ and $\phi^n=\phi*\cdots*\phi$ (n times).

c) If $\mathcal{X}=\Lambda$ or $\mathcal{C}(Q_*)$, then (1) implies that each T_t is a continuous operator on $\mathcal{X}$.

1.3. Definition. A semigroup $(\bar{T}_t)_{t\geqq 0}$ of linear operators on $\mathcal{X}$ is *degenerated* if there is a semigroup $(S_t)_{t\geqq 0}$ of linear operators on $\mathcal{C}(Q)$ such that, for any $f_1, \cdots, f_n$ in $\mathcal{C}(Q)$,

(3) $$\bar{T}_t[(I_1 f_1)*\cdots*(I_1 f_n)]=I_n[(S_t f_1)*\cdots*(S_t f_n)] \qquad \text{for any } t\geqq 0.$$

1.4. Remark. If a semigroup $(\bar{T}_t)_{t\geqq 0}$ associated with some Markov process X with state space Q_* is degenerated, then X is equivalent to the n independent copies of some Markov process with state space Q on each Q_n.

§2. A characterization.

2.1. Theorem. Let $(\bar{T}_t)_{t\geqq 0}$ be a semigroup of bounded operators on $C_0(Q_*)$ and $(\bar{T}_t^*)_{t\geqq 0}$ its dual semigroup on $C_0(Q_*)'$.

(i) The branching property (2) for $(T_t)_{t\geqq 0}$ is equivalent to the condition: if μ and ν are in $C_0(Q_*)'$ and if $\mu*\nu$ is in $C_0(Q_*)'$, then,

(1*) $$\bar{T}_t^*(\mu*\nu)=(\bar{T}_t^*\mu)*(\bar{T}_t^*\nu) \qquad \text{for any } t\geqq 0.$$

(ii) The interaction property (1) for $(\bar{T}_t)_{t\geqq 0}$ is equivalent to the condition: if m is in $\mathcal{M}(Q)$ and of total mass $\|m\|\leqq 1$, then,

(2*) $$\bar{T}_t^*M^*m=M^*R_1^*\bar{T}_t^*M^*m \qquad \text{for any } t\geqq 0.$$

(iii) $(\bar{T}_t)_{t\geqq 0}$ satisfies both of the interaction and branching properties if and only if it is degenerated.

2.2. Remark. The condition (1*) is equivalent to the relation:

(1*′) $$\bar{T}_t^*\bar{M}^*=\bar{M}^*\bar{T}_t^*$$

on $\{\mu\in C_0'(Q_*)': \|\mu\|<1\}$ where $\bar{M}^*\mu=\sum_{n\geqq 1}\mu^n$ and $\mu^n=\mu*\cdots*\mu$ (n times).

2.3. $\langle\, ,\rangle_{Q_n}$, $\langle\, ,\rangle_{Q_*}$ etc. denote the integrals over the sets Q_n, Q_* etc. here and hereafter.

Lemma. 1) *If μ and ν are in $C_0(Q_*)'$ and if f is $C_0(Q)$ and of* norm<1, *then,*

$$\langle\mu*\nu, Mf\rangle_{Q_*}=\langle\mu, Mf\rangle_{Q_*}\langle\nu, Mf\rangle_{Q_*}.$$

2) *If ϕ and ψ are in $C_0(Q_*)$ and if $m\in C_0(Q)'$ is of* norm<1,

$$\langle M^*m, \phi*\psi\rangle_{Q_*}=\langle M^*m, \phi\rangle_{Q_*}\langle M^*m, \psi\rangle_{Q_*}.$$

3) *Let λ, μ and ν be in $C_0(Q^*)'$. If, for any f in $C(Q)$ of* norm<1, *the equality*

$$\langle\lambda, Mf\rangle_{Q_*}=\langle\mu, Mf\rangle_{Q_*}\langle\nu, Mf\rangle_{Q^*}$$

holds, then,

$$\lambda=u*\nu.$$

4) *Let ϕ, ψ, χ be in $C_0(Q_*)$. If, for any m in $\mathcal{M}(Q)$ of* norm<1, *the equality*

$$\langle M^*m, \chi\rangle_{Q_*}=\langle M^*m, \phi\rangle_{Q_*}\langle M^*m, \psi\rangle_{Q_*}$$

holds, then,

$$\chi=\phi*\psi.$$

Proof. It suffices to note that all the integrals in the statements are absolutely convergent.

2.4. Proof of the theorem. (i) and (ii)

(i) is a known result (cf. S. Watanabe [4] for example) and can be proved similarly as (ii). For (ii), we first assume that $(\bar{T}_t)_{t\geqq 0}$ is an interaction semigroup. Let $p\geqq 1$ and $f_1, \cdots, f_p\in C(Q)$. Then, using 2) of the lemma,

$$\begin{aligned}\langle \bar{T}_t^* M^* m, (I_1 f_1) * \cdots * (I_1 f_p)\rangle_{Q_*} &= \langle M^* m, (\bar{T}_t I_1 f_1) * \cdots * (\bar{T}_t I_1 f_p)\rangle_{Q_*} \\ &= \langle M^* m, \bar{T}_t I_1 f_1\rangle_{Q_*} \cdots \langle M^* m, \bar{T}_t I_1 f_p\rangle_{Q_*} \\ &= \langle M^* R_1^* \bar{T}_t^* M^* m, I_1 f_1 * \cdots * I_1 f_p\rangle_{Q_*}\end{aligned}$$

for any m in $\mathcal{M}(Q)$ of norm <1. Consequently (2*) holds. Conversely, if (2*) holds, then, for any ϕ and ψ in $C_0(Q_*)$ such that $\phi * \chi$ is also in $C_0(Q_*)$, by 2) of the lemma,

$$\begin{aligned}\langle M^* m, \bar{T}_t(\phi * \psi)\rangle_{Q_*} &= \langle M^* R_1^* T_t^* M^* m, \phi * \chi\rangle_{Q*} \\ &= \langle M^* m, \bar{T}_t \phi\rangle_{Q_*} \langle M^* m, \bar{T}_t \phi\rangle_{Q_*}.\end{aligned}$$

Applying 4) of the lemma, we have

$$\bar{T}_t(\phi * \psi) = (\bar{T}_t \phi) * (\bar{T}_t \psi).$$

Hence the proof of (ii) is completed.

2.5. Proof of (iii).

The "if" part is obvious. We assume that (1) and (2) hold for $(\bar{T}_t)_{t \geqq 0}$. It is easy to show that

$$(4) \qquad \langle \mu, (Mf) * (Mf)\rangle_{Q_*} = \sum_{k \geqq 1} (k+1) \langle \mu, I_k R_k M f\rangle_{Q*}$$

for any μ in $C_0(Q_*)'$ and f in $C(Q)$ with $\|f\| < 1$. From (1), (2) and (4), it follows that

$$\sum_{k \geqq 1} (k+1) \langle \mu, \bar{T}_t I_k R_k M f\rangle_{Q_*} = \sum_{j \geqq 1} (j+1) \langle \mu, I_j R_j \bar{T}_t M f\rangle_{Q_*}$$

and that

$$I_j R_j \bar{T}_t I_k R_k = 0 \qquad \text{if } j \neq k.$$

Consequently $\bar{T}_t = \sum_{n \geqq 1} I_n T_t^n R_n$ where $\bar{T}_t^n = R_n T_t I_n$, which is equal to $T_t^1 \otimes \cdots \otimes T_t^1$ on $C(Q_n)$ by the interaction property (1). Hence we have proved that $(\bar{T}_t)_{t \geqq 0}$ is degenerated.

§ 3. Non-interacting part.

3.1. Theorem. For any interaction semigroup $(\bar{T}_t)_{t \geqq 0}$, $\bar{T}_t^0 = \sum_{p \geqq 1} I_p R_p \bar{T}_t I_p R_p$ forms a degenerated semigroup; more precisely $R_p \bar{T}_t I_p = T_t^0 \otimes \cdots \otimes T_t^0$ with $T_t^0 = R_1 \bar{T}_t I_1$.

3.2. Definition. The semigroup $(\bar{T}_t^0)_{t \geqq 0}$ is called the *non-interacting part* of the interaction semigroup $(\bar{T}_t)_{t \geqq 0}$.

3.3. The theorem follows immediately from the lemma: If we define a function $F(\alpha, \beta)$ on $\{(\alpha, \beta) \in \boldsymbol{C}^2 : |\alpha| \leqq 1 \text{ and } |\beta| \leqq 1\}$ by $\langle M^*(\alpha m), T_t M(\beta f)\rangle$, for fixed m and f in the open unit ball of $C_0(Q)'$ and $C_0(Q)$ respectively, then it is analytic and satisfies the functional equation

$$F(\alpha, \beta) = \sum_{n \geqq 0} \left(\beta \frac{\alpha F}{\alpha \beta}(\alpha, 0)\right)^n.$$

In Part II, we will give the definition of the Markov processes with interaction, their decomposition and construction and the relation to the branching Markov processes.

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