2. An Application of the Method of Acyclic Models

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The objective of this note is to establish a theorem (Theorem 1) concerning the equivalence between homology theory of a semi-simplicial complex K and singular homology theory of a CW-complex $P(K)$ associated to the complex K. This theorem immediately follows from two theorems (Theorems 2 and 3), and these theorems are both proved by using the powerful method of acyclic models which is established by S. Eilenberg and S. MacLane,[3] recently. Thus the CW-complex $P(K)$ may be regarded as a standard geometric realization of abstract semi-simplicial complex K from the point of view of homology.

1. **Preliminaries.** In this section, we summarize some notations and definitions used in the sequel.

Let K be a semi-simplicial (abbreviated: s. s.) complex, i. e., K be a collection of elements $\{\sigma\}$ called simplexes together with two functions. The first function associates with each simplex σ an integer $q \geq 0$ called the dimension of σ; we then say that σ is a q-simplex. The second function associates with each q-simplex $\sigma(q > 0)$ of K and with each integer $0 \leq i \leq q$ a $(q-1)$-simplex $\sigma^{(i)}$ called the i th face of σ, subject to the condition

$$[\sigma^{(j)}]^{(i)} = [\sigma^{(i)}]^{(j-1)}$$

for $i < j$ and $q > 1$.

We may pass to lower dimensional faces of σ by iteration. If $0 \leq i_1 < \cdots < i_n \leq q$ then we define inductively

$$\sigma^{(i_1, i_2, \dots, i_n)} = [\sigma^{i_2, \dots, i_n)}]^{(i_1)}.$$

This is a $(q-n)$-simplex. If $0 \leq j_0 < \cdots < j_{q-n} \leq q$ is the set complementary to $\{i_1, \dots, i_n\}$ then we also write

$$\sigma^{(i_1, \dots, i_n)} = \sigma_{(j_0, \dots, j_{q-n})}.$$

We write $[q]$ for the set $(0,1, \dots, q)$ where q is an integer ≥ 0. By a map $\alpha: [i] \to [q]$ $(0 \leq i \leq q)$ will be meant a stricted monotone function, i. e., which satisfies $\alpha(i) < \alpha(j)$ for $0 \leq i \leq j \leq q$. Let $\varepsilon_q^i: [q-1] \to [q]$ denote the map which covers all of $[q]$ except $i(i = 0, \dots, q)$. For a q-simplex σ and a function $\alpha: [i] \to [q]$ $(i < q)$, we denote the i-simplex $\sigma_{(\alpha(0), \dots, \alpha(i))}$ by $\sigma\alpha$, and we make the convention that $\sigma\varepsilon_q = \sigma$ for the identity map $\varepsilon_q: [q] \to [q]$.

The boundary of σ is defined as the chain

$$\partial\sigma=\sum_{i=0}^{q}(-1)^{i}\sigma^{(i)},$$

thus the chain complex $C(K)$ is defined by usual fashion.

A simplicial map $f: K\rightarrow K_1$ of a s.s. complex K into another such complex K_1 is a function which to each q-simplex σ of K assigns a q-simplex $\tau=f(\sigma)$ of K_1 is such a fashion that

$$\tau^{(i)}=f(\sigma^{(i)}),\ i=0,\dots,q.$$

Next, we proceed to the definition of the CW′-complex $P(K)$ associated with a s.s. complex K. In the case where K is the singular complex $S(X)$ of a topological space X, this CW′-complex $P(K)$ is the singular polytope termed by Giever.[4]

Let Δ_q denote the unit ordered euclidean q-simplex ($q\geq 0$), ane for each q-simplex σ of K, let (σ, Δ_q) be the rectilinear q-simplex, whose points are the pairs (σ, r), for every point $r\in\Delta_q$, and whose topology and affine geometry are such that the map $r\rightarrow(\sigma, r)$ is a barycentric homeomorphism. For any face s_i of Δ_q we shall denote the corresponding face of (σ, Δ_q) by (σ, s_i).

Let $R(K)$ be the union of all the (disjoint) simplicial complex (σ, Δ_q), for every $q\geq 0$ and every q-simplex σ of K. It is obvious that the ordering of vertices $d^0,\dots,d^q$ of Δ_q, for each $q\geq 0$, and the maps $r\rightarrow(\sigma, r)$ determine a local ordering (cf. Whitehead,[5] § 19) in $R(K)$.

Let (σ, s_i) and (τ, t_j) be i- and j-dimensional faces of (σ, Δ_m) and (τ, Δ_n) respectively. We define the relation $(\sigma, s_i)\equiv(\tau, t_j)$ if, and only if $i=j$ and $\sigma\alpha=\sigma\beta$, where $\alpha:[i]\rightarrow[m]$, $\beta:[j]\rightarrow[n]$ are defined by $s_i=(d^{\alpha(0)},\dots,d^{\alpha(i)})$ and $t_j=(d^{\beta(0)},\dots,d^{\beta(j)})$.

Let (σ, r_1), (τ, r_2) be points in $R(K)$. We write $(\sigma, r_1)\equiv(\tau, r_2)$ if, and only if, there are equivalence simplexes (σ, s_i), (τ, t_i), such that $r_1\in s_i-\dot{s}_i$, $r_2\in t_i-\dot{t}_i$, and $r_2=B(t_i, s_i)r_1$, where $B(t_i, s_i)$ is the order preserving barycentric map of s_i onto t_i. Obviously $(\sigma, r_1)\equiv(\tau, r_2)$ is an equivalence relation. Let $P(K)$ be the space whose points are these equivalence classes of points in $R(K)$ and which has the identification topology determined by the map $\boldsymbol{p}: R(K)\rightarrow P(K)$, where $\boldsymbol{p}(\sigma, r)$ is the class containing (σ, r). Then, in virtue of Lemma 3 (Whitehead,[5] § 19), $P(K)$ is a CW-complex.

Let $R'(K)$ be the derived complex of $R(K)$, in which each new vertex is placed at the centroid of its simplex. We define a local ordering in $R'(K)$ by placing the centroid of (σ, Δ_n) after the centroid of (τ, Δ_m) if $m<n$. Let $R''(K)$ be the derived complex of $R'(K)$, and a local ordering in $R''(K)$ be the ordering induced by the ordering of $R'(K)$ by the same fashion.

Then it is not difficult to verify that the map $\boldsymbol{p}: R(K)\rightarrow P(K)$ induces the simplicial structure $\boldsymbol{p}R''(K)=P''(K)$ and the definite local ordering in $P''(K)$.

Let $f: K\to L$ be a simplicial map. Then a continuous map $P(f): P(K)\to P(L)$ is uniquely defined by $P(f)P(\sigma, r)=P(f\sigma, r)$ for $(\sigma, r)\in R(K)$. Also it is seen that map $P(f)$ may be regarded as an order-preserving non-degenerated simplicial map $P''(f)$ of $P''(K)$ into $P''(L)$.

2. Statements of Theorems

Let $\mathfrak{K}$ be the category consisting of all s. s. complexes and of all simplicial maps. Then the correspondence $K\to C(K)$, is a covariant functor defined on the category $\mathfrak{K}$ with values in the category $\partial\mathfrak{G}$ of chain complexes and chain mappings.

Let $\mathfrak{A}$ be the category consisting of all topological spaces and all continuous maps. Then the correspondence $X\to S(X)$ is a covariant functor $S:\mathfrak{A}\to\mathfrak{K}$.

Let $\mathfrak{P}$ be the category consisting of all simplicial polytopes with the weak topology and with the definite local ordering and of all order-preserving non-degenerated simplicial maps. Then $P'': \mathfrak{K}\to\mathfrak{P}$ is a covariant functor.

Furthermore, we shall consider two functors $C_0:\mathfrak{P}\to\partial\mathfrak{G}$ and $S_0:\mathfrak{P}\to\mathfrak{K}$. For any simplicial polytope $Q\in\mathfrak{P}$, since Q has the definite local ordering it naturally defines a s. s. complex, and $C_0(Q)$ is the chain complex of this s. s. complex. $S_0(Q)$ is the singular complex of Q considering to be a topological space.

Now we can state the theorems.

Theorem 1. *Two covariant functors C, $CSP: \mathfrak{K}\to\partial\mathfrak{G}$ are equivalent i.e., there exist natural transformations $\lambda: C\to CSP$ and $\mu: CSP\to C$ such that $\mu\lambda(K): C(K)\to C(K)$ and $\lambda\mu(K): CSP(K)\to CSP(K)$ are both chain homotopic to the identities, for all $K\in\mathfrak{K}$.*

Theorem 2. *Two covariant functors C, $C_0P'': \mathfrak{K}\to\partial\mathfrak{G}$ are equivalent in the sense of Theorem 1.*

Theorem 3. *Two covariant functors C_0, $CS_0: \mathfrak{P}\to\partial\mathfrak{G}$ are equivalent in the sense of Theorem 1.*

Theorem 1 is an obvious consequence of Theorems 2 and 3, since $P(K)$ and $P''(K)$ coincide as topological spaces. Theorems 2 and 3 are proved in the next two sections.

We shall remark that Theorems 2 and 3 are generalization of Theorems II and V in the Reference 4).

3. Proof of Theorem 2

Let $K_0[m]$ be an m-dimensional s. s. complex defined as following. For each integer q, $0\leq q\leq m$, q-simplex of $K_0[m]$ is any function $\alpha: [q] \to [m]$. The i-face $\alpha^{(i)}$ of $\alpha (i = 0, 1, \ldots, q)$ is defined as the composite map $\alpha \varepsilon_q^i$. Then, for each map $\beta: [i] \to [q]$ $(i \leq q)$ i-simplex $\alpha\beta$ in the notation of §1 is the composite map

$$\alpha\beta : [i] \to [m] .$$

Let $\mathfrak{M}$ be the collection of s. s. complex $K_0[m]$ for all integer $m \geq 0$.

In virtue of Theorem II,[3] Theorem 2 is established if we show that for all m, $n \geq 0$, $H_n(K_0[m]) = 0 = H_n(SPK_0[m])$, and functors C_n, $C_nP'' : \mathfrak{K} \to \mathfrak{G}$ are representable with respect to the models $\mathfrak{M}$, for all $n \geq \upsilon$, where $\mathfrak{G}$ is the category of all abelian groups and homomorphisms.

Since $H_n(K_0[m]) = H_n(\varDelta_m)$ and $H_n(SPK_0[m]) = H_n(S_0\varDelta_m)$, it is obvious that $H_n(K_0[m]) = 0 = H_n(SPK_0[m])$.

Let $K \in \mathfrak{K}$ and let $\sigma \in K$ be any n-simplex. Then we define a simplicial map $\phi_\sigma : K_0[n] \to K$ by $\phi_\sigma(\alpha) = \sigma\alpha$ for $\alpha \in K_0[m]$. Define a map $\varPsi : C_n(K) \to \tilde{C}_n(K)$ by $\varPsi(\sigma) = (\phi_\sigma, \varepsilon_n)$, where ε_n is the unique n-simplex of $K_0[n]$. (For the definition of $\tilde{C}_n(K)$, see the Reference 3), §2.) Then it is easily verified that $\varPsi$ is a natural transformation and this provides the representation of C_n.

Next, let $\xi \in P''(K)$ be any n-simplex. Then ξ is an image $P(\sigma, s)$, where σ is an n-dimensi n-simplex of K and s is an n-simplex of the second derived complex $\varDelta_n''$ of $\varDelta_n$. Such (σ, s) is unique. Define a map $\varPsi : C_nP(K) \to (\widetilde{C_nP})(K)$ by $\varPsi(\xi) = (\phi_\sigma, P(\varepsilon_n, s))$, where $P: R(K_0[m]) \to P(K_0[n])$ is the identification map. This yields a representation of the functor C_nP. Thus the proof of Theorem 2 is completed.

4. Proof of Theorem 3

It is necessary to consider another functor $S_0^* : \mathfrak{P} \to \mathfrak{K}$ defined as following. For each $Q \in \mathfrak{P}$ we define $S_0^*(Q)$ as the subcomplex of $S_0(Q)$ which composed of n-simplexes T such that $T(\varDelta_n)$ is contained in an open star st v of some vertex v of Q.

By (Eilenberg and Steenrod,[1] p. 207), it is easily seen that CS_0, $CS_0^* : \mathfrak{P} \to \partial\mathfrak{G}$ are chain homotopic. Let $\mathfrak{M}$ be the collection of all objects Q of $\mathfrak{P}$ such that Q is contractible to a point on itself. For any $Q \in \mathfrak{M}$, it is well known that $H_n(Q) = 0$ and $H_n(S_0^*(Q)) = H_n(S_0(Q)) = 0$.

Next we show that functors C_0, CS_0^* ; $\mathfrak{P} \to \partial\mathfrak{G}$ are representable with respect to $\mathfrak{M}$ for all dimensions. For C_0 it is obvious.

For any n-simplex $T \in S_0^*(Q)$, $(Q \in \mathfrak{P})$, there are finite vertices v_i such that st $v_i \supset T(\varDelta_n)$, and such vertices form a simplex of Q. Let $v(T)$ be the first vertex of this simplex. Let $M(T)$ be the subcomplex of Q which composed of simplexes with $v(T)$ as the vertex and all their faces. Let $\phi_T: M(T)\ Q\rightarrow$be the inclusion map. Then $M(T)$ belongs to model $\mathfrak{M}$ and map ϕ_T is a map of $\mathfrak{P}$. Define $\Psi(T)=(\phi_T, T')$, where $T': \Delta_n\rightarrow M(T)$ is defined by $T: \Delta_n\rightarrow Q$. Then Ψ yields a representation of CS_0^*. Thus, as in § 2, by Theorem II,[3] C_0 and CS_0^* are chain homotopic, and so C_0 and CS_0 are equivalent, this completes the proof of Theorem 3.

References

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