# 1. *Theory of Path Structure.* I

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1. **Introduction.** The present note is a preliminary report of the paper which will be published elsewhere under the same title.

Veblen[1] constructed a descriptive geometry in terms of "point" and "order" undefined, and on the other hand Prenowitz[2] formulated descriptive geometries as multigroups. Our aim is to show, making use of the Prenowitz's algebraic idea, that the way found by Veblen leads up directly to the fertile land of geometries of manifolds. The geometry of manifold which we are going to construct may be considered as an abstraction of the geometry of paths which was also inaugulated by Veblen with Eisenhart.[3] But a larger part of the studies included in the present paper is concerned to so-called flat manifolds. We shall expect that sooner or later our theory will extend to more general cases.

2. **Concept of Path Structure.** Suppose that there is a non-empty set $S$ whose elements are called *points*. We denote points by small Latin letters, and sets of points (subsets of $S$) will be designated by capital Latin letters. We write $A \approx B$ if $A \cap B \neq 0$.

We now assume that there is an operation —, which associates a certain set of points $a-b$ called the *difference* of $a$ and $b$ with each ordered pair of points $a, b$. The set union $\bigcup \{a-b \mid a \approx A, b \approx B\}$ for non-empty $A, B$ is denoted $A-B$; we define $A-0=0-A=0$ for an arbitrary $A$. The set of all $x$ for which $x-b \approx a$ is designated by $a+b$, and it is called the *sum* of $a$ and $b$. If $A, B \neq 0$, then $A+B$ is the set union $\bigcup \{a+b \mid a \approx A, b \approx B\}$. We define further $A+0=0+A=0$ for an arbitrary $A$.

The set $S$ thus furnished with the operation — will be denoted $(S; -)$. We assume that $(S; -)$ satisfies the followings:

I. *Idempotent Law*: $a-a=a$.

II. *Commutative Law*: $a+b=b+a$.

III. *Absorptive Law*: $a+(a-b) \subset a-b$.

IV. *Reductive Law*: *If* $a+b \approx a+c$ *and* $b \neq c$, *then we have either* $b \approx a+c$, $b \neq c$ *or* $c \approx a+b$, $b \neq c$.

V. *If* $a-b \neq 0$, *then* $a+b \neq 0$.

Then $(S; -)$ is referred to as a *path structure*. From now on our investigations will be carried out in the path structure $(S; -)$.

A directed pair $(a, b)$ is called *subtractive* if $a-b \neq 0$. A directed set pair $(A, B)$ is called subtractive if every $(x, y) \mid x \approx A$, $y \approx B$ is subtractive. A set $A$ is called subtractive if $(A, A)$ is subtractive. A subtractive set $A$ is called pseudo-open if for any $p, q$ of $A$ there is $x$ such that $x \approx q-p$ and $q+x \subset A$.

A directed point pair $(a, b)$ is called *additive* if either $a=b$ or $a \neq b$, $a+b \neq 0$, $x, y \approx a+b$ and $x \neq y$ give rise to $a \approx x-y$ or $a \approx y-x$. We can show that *if $(a, b)$ is additive, then $(b, a)$ is also additive, and the subtractivity of $(a, b)$ implies the additivity of* $(a, b)$. The additivities of set pairs and sets will be defined as in the case of the subtractivities.

Let $(a, b)$ be additive. Then $a+b$ is called an *open segment*. We define $a \oplus b = a \cup (a+b) \cup b$, and call it a *closed segment*. If $a \cup b$ is subtractive, then each of $a+b$ and $a \oplus b$ is called extensible. Next suppose that $p \approx a+b$, then $p-(p+b)=a/p$ is termed a *ray* emanating from $p$.

We say that a set $A$ *supports* a ray $a/p$ if $A$ contains $p$ and $x$ such that $a/p = x/p$. Let $A, B$ be any sets. If every ray which emanates from $p$ and at the same time supported by $B$ is supported by $A$, then $A$ is said to support $B$ round $p$. If $A$ supports $B$ round every point of $A \cap B$, then we say that $A$ supports $B$. If $A$ and $B$ support each other round $p$, then we write $A \approxeq B \pmod p$. If $A \approxeq B \pmod p$ for every $p \approx A \cap B$, then we denote this by $A \approxeq B$, and say that $A$ and $B$ have a *perfect intersection*. If $A \approxeq B$ and $A \subset B$, then $A$ is said to be *perfect in* $B$.

A chain of segments (open or closed) is a finite sequence of segments such that the interiors of any two neighbouring segments have a perfect intersection. A collection of segments is called connected if any two segments of the collection are connected by a chain of segments of the collection. The set union of all the segments of a maximal connected collection of segments is termed a *path* generated by any segment out of the collection. The *extremity* of a path is the set of at most two points which bound the path (the precise definition is now omitted).

In the above argument if only open segments are taken into consideration, then we get a *cyclic path* which has the empty extremity. Any path contains one and only one cyclic path which is the set difference of the path and its extremity.

3. **Linear Structure.** Suppose that $(S; -)$ satisfies

*Line Postulate: Let $a \cup b$ be subtractive, then $(a+b)-x$ remains the same so far as $x$ ranges over $a+b$.*

Then $(S; -)$ is said to form a *linear structure*. And any path in a linear structure is termed a *line*. We define

$$a \vee b = a \cup b \cup (a+b) \cup (a-(a+b)) \cup (b-(a+b)) .$$

The set $a \vee b$ has also its extremity composed of at most two points bounding $a \vee b$ (the precise definition is omitted). We can prove: *If a line is generated by a segment $a+b$, then $a \vee b$ is identical with the line, and the extremity of $a \vee b$ is identical with the extremity of the line. If each of $p, q$ on a line is not contained in the non-empty extremity of the line, then $p \cup q$ is subtractive and the line is identical with $p \vee q$.*

We only notice that there are open lines and closed lines. The description of various properties of lines are omitted.

**4. Theory of Convexes.** For an arbitrary $A$ the set $A+A$ is termed the *self-sum* of $A$. We can define the successive self-sums of $A$. If $A$ and its successive self-sums up to the $(n-1)$th are each additive, then we say that $A$ is *n-additive*; if all the successive self-sums are additive, then $\infty$-additive. If $A+A \subset A$, then $A$ is called *convex* provided that $A$ is additive.

Let a finite set $A$ composed of $n$ points $(n \geqq 2)$ be $(n-1)$ additive, and let $(x+y)+z = x+(y+z)$ for any points $x, y, z$ of the $(n-1)$th self-sum of $A$, then $A$ is said to form an *associative point system*. And even if $A$ is not finite, it is also said to form an associative point system if every finite subset of $A$ forms an associative point system defined above. Further a single point $a$ is said to form an associative point system.

Let $A = a_1 \cup \dots \cup a_n$ form an associative point system. Then we can prove that each of $a_1 + \cdots + a_n$ and $a_1 \oplus \cdots \oplus a_n$ is independent of the order in which the letters $a_i$ appear, hence we can write $a_1 + \cdots + a_n = \sum A$ and $a_1 \oplus \cdots \oplus a_n = \sum \oplus A$. We can prove: *$\sum \oplus A$ is the maximal convex set which includes $A$, and $\sum A$ is the maximal pseudo-open set included in $\sum \oplus A$.* $\sum \oplus A$ is the convex generated by $A$ and $\sum A$ is the open convex (or the interior of $\sum \oplus A$) generated by $A$. The set difference $\partial A = \sum \oplus A - \sum A$ is the boundary of $\sum \oplus A$. We can clarify the constitution of $\sum \oplus A$ in detail. The method to do this is quite different from that of Weyl[4] because we do not suppose that $\sum \oplus A$ is included in an affine space. If $A$ is independent, then $\sum \oplus A$ is a simplex.

In case $A$ is infinite we can define $\sum \oplus A$ as the set union of all finite $\sum A' \mid A' \subset A$, and call it also the convex generated by $A$. If $A$ is independent, the $\sum \oplus A$ is an infinite simplex. What is remarkable for an infinite simplex is that it has no interior, that is, the maximal pseudo-open set contained in the simplex.

**5. Linear and Associative Structures.** We treat here linear structures which satisfy various types of associative laws. A polygon $[pp_1 \ldots p_n q]$ is the set union of segments $p \oplus p_1, \ldots, p_n \oplus q$. A set $A$ any two points of which are connected by a polygon included in $A$ is called *arcwise connected*, and if each connecting polygon is composed of only extensible segments, then $A$ is called *subtractively arcwise connected.*

*D-structure.* Let $(S; -)$ satisfy

*Associative Law*: $(a+b)+c = a+(b+c)$,

then $(S; -)$ is referred to as a $D$-structure. Any subtractively arcwise connected set is called a *component* of $(S; -)$ if it is maximal. Then we can prove: *Every component of* $(S; -)$ *is a descriptive geometry.*

*DS-structure.* Let $(S; -)$ be linear and satisfy

$\oplus$*-associative Law*: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,

then $(S; -)$ is referred to as a $DS$-structure. A component of $(S; -)$ is defined as in the case of the $DS$-structure. Then we have: *If a component of DS-structure is not composed of a single line, then it is either a descriptive geometry or a spherical geometry.*

*P-structure.* We define

$$a \nabla b = (a+b) \cup (a-(a+b)) \cup (b-(a+b)),$$
$$A \nabla B = \bigcup \{a \nabla b \mid a \approx A,\ b \approx B\}.$$

Let $(S; -)$ be linear and satisfy

$\nabla$*-associative Law*: $(a \nabla b) \nabla c = a \nabla (b \nabla c)$,

then $(S; -)$ is referred to as a $P$-structure. A maximal arcwise connected set of $(S; -)$ is termed a component of $(S; -)$. Then we can prove: *Every component of* $(S; -)$ *is a projective geometry.*

We notice that in the $P$-structure it is not always the case that every line is cyclic.

*SP-structure.* We say that $(S; -)$ is semi-symmetric if $a-b \neq 0$ implies $b-(a+b) \neq 0$. Let $(S; -)$ be linear, semi-symmetric and satisfy

$\vee$*-associative Law*: $(a \vee b) \vee c = a \vee (b \vee c)$,

then $(S; -)$ is referred to as a $SP$-structure. A component of $SP$-structure is defined as in the case of the $P$-structure. Then we have: *Every component of* $(S; -)$ *is a spherical geometry if it is not a projective geometry.*

*DSE-structure.* Let $(S; -)$ be linear and satisfy the followings:

*Restricted Associative Law*: *If the pairs* $(a, b)$, $(b, c)$, $(c, a)$ *and* $(a+b, c)$ *are all additive, then* $(a+b)+c = a+(b+c)$.

*Plane Postulate*: *Let* $a+b+c$ *be a triangle* (2*-simplex*), *then* $(a+b+c)-p$ *remains the same so far as* $p$ *ranges over* $a+b+c$.

*Condition*: *If the pairs* $(p, a)$ *and* $(p, b)$ *are non-additive, then the pairs* $(p, a-b)$ *and* $(p, b-a)$ *are both non-additive* ($(A, B)$ is called non-additive if there are no $x \approx A$ and $y \approx B$ such that $(x, y)$ is additive).

Then $(S; -)$ is referred to as a *DSE*-structure. A maximal subtractively arcwise connected set is termed a component of $(S; -)$. Then we can prove: *Every component of* $(S; -)$ *is any one of geometries, a descriptive or a spherical or a projective.*

**6. Region System.** We say that $X$ is enclosed in $Y$ if the self-sum of $X$ is included in $Y$. If the $n$th self-sum is included in $Y$, then, $X$ is said to be $n$-enclosed in $Y$. A sequence of pseudo-open sets $A_1, \dots, A_n$ is said to form a chain if any two neighbouring sets have a perfect intersection. And a polygon $[ap_1 \dots p_{n-1}b]$ ($a$ segment may be considered as a polygon) is said to be confined in this chain if $a \oplus p_1 \subset A_1, \dots, p_{n-1} \oplus b \subset A_n$.

A collection $\mathfrak{R}$ of pseudo-open sets is referred to as a region system if it subjects to the conditions:

$R$ 1. *If* $A \in \mathfrak{R}$ *supports* $B \in \mathfrak{R}$ *round a point, then* $A \approx B$.

$R$ 2. *Let* $A, B \in \mathfrak{R}$ *and* $A \approx B$. *Then any segment* $a \oplus b \subset A \cap B$ *is confined in a chain of sets of* $\mathfrak{R}$, *each link of which is enclosed in* $A \cap B$.

And every set of $\mathfrak{R}$ is called a region of $\mathfrak{R}$.

Any subtractive set $X$ is called $\mathfrak{R}$-*open* if every segment $a \oplus b \subset X$ is confined in a chain in $\mathfrak{R}$, each link of which is enclosed and perfect in $X$. Region systems $\mathfrak{A}$ and $\mathfrak{B}$ are called equivalent if every region of $\mathfrak{A}$ is $\mathfrak{B}$-open and every region of $\mathfrak{B}$ is $\mathfrak{A}$-open. A region system $\mathfrak{R}'$ is called a *refinement* of a region system $\mathfrak{R}$ if for each region $A$ of $\mathfrak{R}'$ there is a set $X$ of $\mathfrak{R}$ such that $A \subset X$ and any region of $\mathfrak{R}$ is $\mathfrak{R}'$-open.

An $\mathfrak{R}$-open set $A$ is called $\mathfrak{R}$-*normal* if any segment $a \oplus b \mid a, b \approx A$ is confined in a chain in $\mathfrak{R}$ each of the initial and the terminal links of which has a perfect intersection with $A$. If every region of $\mathfrak{R}$ is $\mathfrak{R}$-normal, then $\mathfrak{R}$ is called a *normal region system.* Then we can prove: *The collection of all the* $\mathfrak{R}$-*normal sets forms a normal region system which is equivalent to* $\mathfrak{R}$.

**7. Path Manifold.** If $(S; -)$ has a normal region system $\mathfrak{S}$ such that every segment in $(S; -)$ is confined in a chain of the region of $\mathfrak{S}$, then we say that $(S; -)$ forms a *path manifold* $(S; -, \mathfrak{S})$. If a chain of regions of $\mathfrak{S}$ confines a segment $a \oplus b$ and has the initial link $A$ and the terminal link $B$, then $A$ is said to be transitive to $B$ along $a \oplus b$. If any region of $\mathfrak{S}$ which supports any segment $a \oplus b$ round $a$ is transitive along $a \oplus b$, then $(S; -, \mathfrak{S})$ is called *transitive.* $(S; -, \mathfrak{S})$ is called *connected* if any two regions of $\mathfrak{S}$ is connected by a chain in $\mathfrak{S}$.

If a chain in $\mathfrak{S}$ whose initial link is $A$ confines a polygon $[p \dots q]$, then the polygon is called a route and denoted $A[p \dots q]$.

Let $\mathfrak{A}$ be any refinement of $\mathfrak{S}$. We can define "connectedness" and "route" by using only regions of $\mathfrak{A}$. Let $A[p\ldots q]$ and $A[p\ldots q]'$ be any two routes each of which is confined in a chain in $\mathfrak{A}$. And suppose that there exists a finite sequence of routes $A[p\ldots q]$, $A[p\ldots q]_1, \ldots, A[p\ldots q]'$ such that any two neighbouring routes are confined in the same chain in $\mathfrak{A}$ whose initial link has a perfect intersection with $A$. Then we say that $A[p\ldots q]$ and $A[p\ldots q]'$ are *$\mathfrak{A}$-homotopic* to each other. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two refinements of $\mathfrak{S}$ such that any two routes which are $\mathfrak{A}$-homotopic are also $\mathfrak{B}$-homotopic and vice versa, then we say that $\mathfrak{A}$-homotopy and $\mathfrak{B}$-homotopy are *equivalent*. If for any refinement $\mathfrak{A}'$ of $\mathfrak{A}$, $\mathfrak{A}$-homotopy is equivalent to $\mathfrak{A}'$-homotopy, then $\mathfrak{A}$-homotopy is called *universal*, and we say that $(S; -, \mathfrak{S})$ has a universal homotopy. If $(S; -, \mathfrak{S})$ is simply connected in the sense of a universal homotopy, then $(S; -, \mathfrak{S})$ is merely called *simply connected.*

If $(S; -, \mathfrak{S})$ is connected, we can consider covering manifolds of $(S; -, \mathfrak{S})$ which are similar to that covering manifolds familiar to us in topology. We can prove: *If $(S; -, \mathfrak{S})$ has a universal homotopy, then it has a connected and simply connected covering manifold.*

If for any point of $(S; -, \mathfrak{S})$ any two regions of $\mathfrak{S}$ which include the point have a perfect intersection, then $(S; -, \mathfrak{S})$ is called non-branched. $(S; -, \mathfrak{S})$ forms a neighbourhood space if $\mathfrak{S}$ is assigned for neighbourhood system, neighbourhoods of $p$ being denoted $U_p, V_p, \ldots$ . $(S; -, \mathfrak{S})$ is called operationally closed if the following condition is satisfied: *If $a, b, c$ are distinct and have neighbourhoods $U_a$, $U_b$, $U_c$ such that any $V_a \subset U_a$, $V_b \subset U_b$ and $V_c \subset U_c$ have respectively points $p, q, r$ satisfying $p \approx q - r$, then $a \approx b - c$.*

If $(S; -, \mathfrak{S})$ is connected and non-branched, then $(S; -, \mathfrak{S})$ is called a *path space*. We can prove: *If the space $(S; -, \mathfrak{S})$ is operationally closed, then it is a Hausdorff space.*

By means of introducing various topological conditions in $(S; -, \mathfrak{S})$, we can specialize $(S; -, \mathfrak{S})$ in various manners. We set forth here a theorem: *If $(S; -, \mathfrak{S})$ is separable and locally compact, then it has a universal homotopy.*

Finally we notice, using a theorem of G. H. C. Whitehead,[5] that the space of geometry of paths is a path space with a convex region system.

8. **Stratified Structure.** We consider here a path structure which contains a certain collection of path submanifolds, and a theory of dimension and also a theory of separation are discussed. Finally the *DSE*-structure is studied as a sort of stratified structures.

**9. Locally Associative Manifold.** $(S; -, \mathfrak{S})$ is always considered to be connected and transitive. If $\mathfrak{S}$ has a refinement $\mathfrak{S}^{(a)}$ such that for any $A \in \mathfrak{S}^{(a)}$ and $a, b, c \approx A$ we have $(a+b)+c = a+(b+c)$, then $(S; -, \mathfrak{S})$ is called a *locally associative manifold.* We can make this manifold a sort of stratified structures which has the same dimension at every point of it. We indicate by $\mathfrak{S}^{(2ea)}$ a refinement of $\mathfrak{S}^{(a)}$ such that each region of $\mathfrak{S}^{(2ea)}$ is 2-enclosed in some one of regions of $\mathfrak{S}^{(a)}$.

If $(S; -, \mathfrak{S})$ is of dimension $\geqq 3$, then we can embed each $A \in \mathfrak{S}^{(2ea)}$ in a spherical geometry. Using this embedding, we can prove: *Let $(S; -, \mathfrak{S})$ of dimension $\geqq 3$ be simply connected in the sense of $\mathfrak{S}^{(2ea)}$-homotopy, and let any two points be combined by at least one path. If there are no distinct paths which intersect at three distinct points and all the paths are closed, then $(S; -, \mathfrak{S})$ forms a spherical geometry.*

$(S; -, \mathfrak{S})$ is called *continuous* if every segment in $(S; -, \mathfrak{S})$ forms a Dedekind's continuum. In the continuous case we can prove: *Let $(S; -, \mathfrak{S})$ of dimension $\geqq 3$ be continuous. If there is $A \in \mathfrak{S}$ such that every path which goes through $a$ and is supported by $A$ round $a$ is closed, then $(S; -, \mathfrak{S})$ forms a spherical geometry or a projective geometry.*

And we can prove: *Let $(S; -, \mathfrak{S})$ of dimension $\geqq 3$ be continuous. If any points are combined by a single open path, then $(S; -, \mathfrak{S})$ forms a descriptive geometry.*

*Line Postulate* in $(S; -, \mathfrak{S})$ asserts: *If $a+b$ is any segment included in any region of $\mathfrak{S}$, then $(a+b)-p$ is left invariant so far as $p$ ranges over $a+b$.*

And *Plane Postulate in* $(S; -, \mathfrak{S})$ asserts: *Let $a+b+c$ be any triangle included in any region of $\mathfrak{S}$, then $(a+b+c)-p$ is left invariant so far as $p$ ranges over $a+b+c$.*

Then we can prove: *Let $(S; -, \mathfrak{S})$ of dimension $\geqq 3$ satisfy the plane postulate, then it satisfies the line postulate. And $(S; -, \mathfrak{S})$ forms a spherical geometry if there are two distinct lines which intersect at two distinct points, and $(S; -, \mathfrak{S})$ forms a descriptive geometry if there is no closed line. Further $(S; -, \mathfrak{S})$ forms a projective geometry if there are no two distinct lines which intersect at two distinct points and all the lines are closed.*

In the continuous case we can prove: *If $(S; -, \mathfrak{S})$ of dimension $\geqq 2$ is continuous and operationally closed, then $(S; -, \mathfrak{S})$ admits the plane postulate if it admits the line postulate.*

Further we have: *Let $(S; -, \mathfrak{S})$ of dimension $\geqq 2$ be continuous and compact, then $(S; -, \mathfrak{S})$ forms either a spherical geometry or a projective geometry.*

## References

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