32. Probabilities on Inheritance in Consanguineous Families. IV

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(Comm. by T. FURUHATA, M.J.A., Feb. 12, 1954)

IV. Ancestors-descendant combinations through an intermediate marriage

1. Ancestor-parent-child combination immediate after a marriage

Suppose that two individuals $A_{\alpha\beta}$ and $A_{\gamma\delta}$ are accompanied by their μth and νth descendants A_{ab} and A_{cd}, respectively, and that these descendants are married and originate themselves an nth descendant $A_{\xi\eta}$. Let the probability of a triple consisting of $(A_{\alpha\beta}, A_{\gamma\delta}; A_{\xi\eta})$ be then designated by

$$\overline{A}_{\alpha\beta}\overline{A}_{\gamma\delta}\varepsilon_{\mu\nu;n}(\alpha\beta, \gamma\delta; \xi\eta).$$

The probability of parents-descendant combination, ε_n, treated in I, § 2, may be regarded to correspond to the *lowest case* $\mu=\nu=0$; in particular, $\varepsilon_1\equiv\varepsilon_{00;1}$ represents nothing but ε. Here we distinguish *four systems* in case of higher generation-numbers μ, ν according to $\mu>0=\nu$, $n=1$ or $\mu=0<\nu$, $n=1$; $\mu>0=\nu$, $n>1$ or $\mu=0<\nu$, $n>1$; $\mu>0$, $\nu>0$, $n=1$; and $\mu>0$, $\nu>0$, $n>1$.

The *first system* will now be treated. By virtue of an evident quasi-symmetry property with respect to $\alpha\beta$, $\gamma\delta$ and μ, ν, it suffices to consider the former. Its defining equation

$$\varepsilon_{\mu0;1}(\alpha\beta, \gamma\delta; \xi\eta)=\sum\kappa_\mu(\alpha\beta; ab)\varepsilon(ab, \gamma\delta; \xi\eta)$$

can be brought into the form

$$\varepsilon_{\mu0;1}(\alpha\beta, \gamma\delta; \xi\eta)=\kappa(\gamma\delta; \xi\eta)+2^{-\mu}C_0(\alpha\beta, \gamma\delta; \xi\eta),$$

where C_0 is defined by

$$C_0(\alpha\beta, \gamma\delta; \xi\eta)=2\sum Q(\alpha\beta; ab)\varepsilon(ab, \gamma\delta; \xi\eta).$$

The values of C_0 are set out as follows:

$C_0(ii, ii; ii)=1-i$, $C_0(ii, ii; ig)=-g$;
$C_0(ii, ik; ii)=\frac{1}{2}(1-i)$, $C_0(ii, ik; ik)=\frac{1}{2}(1-i-k)$,
$C_0(ii, ik; kk)=-\frac{1}{2}k$, $C_0(ii, ik; ig)=-\frac{1}{2}g$,
$C_0(ii, ik; kg)=-\frac{1}{2}g$;
$C_0(ii, kk; ik)=1-i$, $C_0(ii, kk; kk)=-k$,
$C_0(ii, kk; kg)=-g$;
$C_0(ii, hk; ik)=\frac{1}{2}(1-i)$, $C_0(ii, hk; hk)=-\frac{1}{2}(h+k)$;
$C_0(ij, ii; ii)=\frac{1}{2}(1-2i)$, $C_0(ij, ii; ij)=\frac{1}{2}(1-2j)$,

*) I-III, Proc. Japan Acad. **30** (1954), 42-52.

$C_0(ij, ii; ig) = -\frac{1}{2}g;$

$C_0(ij, ij; ii) = \frac{1}{4}(1-2i),$ $\quad C_0(ij, ij; ij) = \frac{1}{2}(1-i-j),$

$C_0(ij, ij; ig) = -\frac{1}{2}g;$

$C_0(ij, ik; ij) = \frac{1}{4}(1-2j),$ $\quad C_0(ij, ik; ik) = \frac{1}{4}(1-2i-2k),$

$C_0(ij, ik; jk) = \frac{1}{4}(1-2j);$

$C_0(ij, kk; ik) = \frac{1}{2}(1-2i).$

The values of $C_0(\alpha\beta, \gamma\delta; \xi\eta)$ are independent of $\alpha\beta$, provided $A_{\alpha\beta}$ contains no gene in common with $A_{\gamma\delta}$ as well as $A_{\xi\eta}$. It further satisfies the following relations:

$$\sum C_0(\alpha\beta, \gamma\delta; ab) = \sum \bar{A}_{ab} C_0(ab, \gamma\delta; \xi\eta) = 0, \quad \sum \bar{A}_{ab} C_0(\alpha\beta, ab; \xi\eta) = Q(\alpha\beta; \xi\eta).$$

2. Ancestor-parent-descendant combination distant after a marriage

We consider the *second system* with $\mu > 0 = \nu$, $n > 1$. The reduced probability is then defined by an equation

$$\varepsilon_{\mu 0;n}(\alpha\beta, \gamma\delta; \xi\eta) = \sum \varepsilon_{\mu 0;1}(\alpha\beta, \gamma\delta; ab)\kappa_{n-1}(ab; \xi\eta)$$

which leads to an expression

$$\varepsilon_{\mu 0;n}(\alpha\beta, \gamma\delta; \xi\eta) = \kappa_n(\gamma\delta; \xi\eta) + 2^{-\mu-n+1} C(\alpha\beta, \gamma\delta; \xi\eta).$$

The values of a quantity defined by

$$C(\alpha\beta, \gamma\delta; \xi\eta) = 2\sum C_0(\alpha\beta, \gamma\delta; ab)\kappa(ab; \xi\eta)$$

are set out in the following lines:

$C(ii, ii; ii) = i(1-i),$ $\quad C(ii, ii; ig) = g(1-2i),$

$C(ii, ii; gg) = -g^2,$ $\quad C(ii, ii; fg) = -2fg;$

$C(ii, ik; ii) = i(1-i),$ $\quad C(ii, ik; ik) = k(1-2i),$

$C(ii, ik; kk) = -k^2,$ $\quad C(ii, ik; ig) = g(1-2i),$

$C(ii, ik; kg) = -2kg;$

$C(ii, kk; ii) = i(1-i),$ $\quad C(ii, kk; ik) = k(1-2i),$

$C(ii, kk; kk) = -k^2,$ $\quad C(ii, kk; ig) = g(1-2i),$

$C(ii, kk; kg) = -2kg;$

$C(ii, hk; ik) = k(1-2i),$ $\quad C(ii, hk; hk) = -2hk;$

$C(ij, ii; ii) = \frac{1}{2}i(1-2i),$ $\quad C(ij, ii; ij) = \frac{1}{2}(i+j-4ij),$

$C(ij, ii; jj) = \frac{1}{2}j(1-2j),$ $\quad C(ij, ii; ig) = \frac{1}{2}g(1-4i),$

$C(ij, ii; jg) = \frac{1}{2}g(1-4j);$

$C(ij, ij; ii) = \frac{1}{2}i(1-2i),$ $\quad C(ij, ij; ij) = \frac{1}{2}(i+j-4ij),$

$C(ij, ij; ig) = \frac{1}{2}g(1-4i);$

$C(ij, ik; ij) = \frac{1}{2}(i+j-4ij),$ $\quad C(ij, ik; ik) = \frac{1}{2}k(1-4i),$

$C(ij, ik; jk) = \frac{1}{2}k(1-4j);$

$C(ij, kk; ii) = \frac{1}{2}i(1-2i),$ $\quad C(ij, kk; ij) = \frac{1}{2}(i+j-4ij),$

$C(ij, kk; ik) = \frac{1}{2}k(1-4i).$

Several identities are satisfied by C; for instance, we have

$$\sum C(\alpha\beta,\gamma\delta;ab)=\sum\overline{A}_{ab}C(ab,\gamma\delta;\xi\eta)=0,\quad \sum\overline{A}_{ab}C(\alpha\beta,ab;\xi\eta)=Q(\alpha\beta,\xi\eta),$$
$$\sum Q(\alpha\beta;ab)E(ab,\gamma\delta;\xi\eta)=\sum Q(\alpha\beta;ab)C(ab,\gamma\delta;\xi\eta)=\tfrac{1}{2}C(\alpha\beta,\gamma\delta;\xi\eta).$$

3. Ancestors-sdecendant combination immediate after a marriage

The defining equation of the *third system* with $\mu,\ \nu>0,\ n=1$:

$$\varepsilon_{\mu\nu;1}(\alpha\beta,\gamma\delta;\xi\eta)=\sum\kappa_\mu(\alpha\beta;ab)\kappa_\nu(\gamma\delta;cd)\varepsilon(ab,cd;\xi\eta)$$

leads to an expression

$$\varepsilon_{\mu\nu;1}(\alpha\beta,\gamma\delta;\xi\eta)=\overline{A}_{\xi\eta}+2^{-\mu}Q(\alpha\beta;\xi\eta)+2^{-\nu}Q(\gamma\delta;\xi\eta)+2^{-\mu-\nu}D_0(\alpha\beta,\gamma\delta;\xi\eta).$$

The values of a quantity defined by

$$D_0(\alpha\beta,\gamma\delta;\xi\eta)=4\sum Q(\alpha\beta;ab)Q(\gamma\delta;cd)\varepsilon(ab,cd;\xi\eta)$$
$$=2\sum Q(\alpha\beta;ab)C_0(\gamma\delta,ab;\xi\eta)$$

are set out in the following lines:

$D_0(ii,ii;ii)=(1-i)^2$, $\quad D_0(ii,ii;ig)=-2g(1-i)$,
$D_0(ii,ii;gg)=g^2$, $\quad D_0(ii,ii;fg)=2fg$;
$D_0(ii,ik;ii)=\frac{1}{2}(1-i)(1-2i)$, $\quad D_0(ii,ik;ik)=\frac{1}{2}(1-i-3k+4ik)$,
$D_0(ii,ik;kk)=-\frac{1}{2}k(1-2k)$, $\quad D_0(ii,ik;ig)=-\frac{1}{2}g(3-4i)$,
$D_0(ii,ik;kg)=-\frac{1}{2}g(1-4k)$;
$D_0(ii,kk;ii)=-i(1-i)$, $\quad D_0(ii,kk;ik)=1-i-k+2ik$,
$D_0(ii,kk;ig)=-g(1-2i)$;
$D_0(ii,hk;ik)=\frac{1}{2}(1-i-2k+4ik)$, $\quad D_0(ii,hk;hk)=-\frac{1}{2}(h+k-4hk)$;
$D_0(ij,ij;ii)=\frac{1}{4}(1-2i)^2$, $\quad D_0(ij,ij;ij)=\frac{1}{2}(1-2i-2j+4ij)$,
$D_0(ij,ij;ig)=-g(1-2i)$;
$D_0(ij,ik;ij)=\frac{1}{4}(1-2i-4j+8ij)$, $\quad D_0(ij,ik;jk)=\frac{1}{4}(1-2j-2k+8jk)$.

The quantity $D_0(\alpha\beta,\gamma\delta;\xi\eta)$ satisfies, besides a symmetry relation with respect to $\alpha\beta$ and $\gamma\delta$, further identities

$$\sum D_0(\alpha\beta,\gamma\delta;ab)=\sum\overline{A}_{ab}D_0(ab,\gamma\delta;\xi\eta)=0,$$
$$\sum Q(\alpha\beta;ab)D_0(\gamma\delta,ab;\xi\eta)=\tfrac{1}{2}D_0(\alpha\beta,\gamma\delta;\xi\eta).$$

4. Ancestors-descendant combination distant after a marriage

The reduced probability for the *last generic system* with $\mu,\ \nu>0$, $n>1$ is given by an equation

$$\varepsilon_{\mu\nu;n}(\alpha\beta,\gamma\delta;\xi\eta)=\sum\varepsilon_{\mu\nu;1}(\alpha\beta,\gamma\delta;ab)\kappa_{n-1}(ab;\xi\eta)$$

whence follows

$$\varepsilon_{\mu\nu;n}(\alpha\beta,\gamma\delta;\xi\eta)=\overline{A}_{\xi\eta}+2^{-n+1}\{2^{-\mu}Q(\alpha\beta;\xi\eta)+2^{-\nu}Q(\gamma\delta;\xi\eta)\}.$$

In fact, there holds identically a relation

$$\sum D_0(\alpha\beta,\gamma\delta;ab)Q(ab;\xi\eta)=\sum D_0(\alpha\beta,\gamma\delta;ab)\kappa(ab;\xi\eta)=0.$$

It is in passing shown here that there holds a further identity

$$\sum Q(\alpha\beta;ab)C(\gamma\delta,ab;\xi\eta)=0.$$

Asymptotic behaviors of $\varepsilon_{\mu\nu;n}$ as μ, ν, or n tends to ∞ are readily derived. There hold, in fact, the limit equations

$$\lim_{\mu\to\infty} \varepsilon_{\mu\nu;n}(\alpha\beta, \gamma\delta; \xi\eta)=\kappa_{\nu+n}(\gamma\delta; \xi\eta), \quad \lim_{\nu\to\infty} \varepsilon_{\mu\nu;n}(\alpha\beta, \gamma\delta; \xi\eta)=\kappa_{\mu+n}(\alpha\beta; \xi\eta),$$

and

$$\lim_{n\to\infty} \varepsilon_{\mu\nu;n}(\alpha\beta, \gamma\delta; \xi\eta)=\bar{A}_{\xi\eta},$$

which are valid for any values of μ, ν, n with $\nu\geqq 0$, $n\geqq 1$, with $\mu\geqq 0$, $n\geqq 1$, and with $\mu\geqq 0$, $\nu\geqq 0$, respectively.