52. *Probabilities on Inheritance in Consanguineous Families. VIII*

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VIII. Mother-descendants combinations through several consanguineous marriages (Continuation)

3. General mother-descendants combinations through several consanguineous marriages

In the present section we consider the problems which correspond to those discussed in VI, § 3, but we now suppose that there exist two descendants instead of one. The reduced probability in consideration is then defined by

$$\kappa_{l\,|(\mu\nu;\,n)_t\,|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sum\kappa_{l\,|(\mu\nu;\,n)_t}(\alpha\beta;\,ab)\kappa_{\mu\nu}(ab;\,\xi_1\eta_1,\,\xi_2\eta_2)\quad(\mu\nu=\mu_{t+1}\nu_{t+1}).$$

In case $\mu=\nu=1$, we get the following results:

$$\kappa_{l\,|(\mu\nu;\,1)_t\,|11}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sigma(\xi_1\eta_1,\,\xi_2\eta_2)+4(u_t+w_t)\mathfrak{X}(\xi_1\eta_1,\,\xi_2\eta_2)$$
$$+2^{-l+1}\{2^{-t}\Lambda_tU(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)+(v_t+2w_t)\mathfrak{Y}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\},$$
$$\kappa_{l\,|(\mu\nu;\,n)_t\,|11}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sigma(\xi_1\eta_1,\,\xi_2\eta_2)+2^{-l-N_t+1}\Lambda_tU(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\quad\text{for } n_t>1.$$

In case $\mu=1<\nu$ we get the following results:

$$\kappa_{l\,|(\mu\nu;\,1)_t\,|1\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sigma_{1\nu}(\xi_1\eta_1,\,\xi_2\eta_2)+2^{-\nu+1}(u_t+w_t)\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\,\xi_2\eta_2)$$
$$+2^{-l-t}\Lambda_t\{\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\,\xi_1\eta_1)+2^{-\nu+1}V(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\}$$
$$+2^{-l-\nu}(v_t+2w_t)S(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2),$$
$$\kappa_{l\,|(\mu\nu;\,n)_t\,|1\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sigma_{1\nu}(\xi_1\eta_1,\,\xi_2\eta_2)$$
$$+2^{-l-N_t}\Lambda_t\{\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\,\xi_1\eta_1)+2^{-\nu+1}V(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\}\quad\text{for } n_t>1.$$

In case $\mu,\ \nu>1$, we get the following results:

$$\kappa_{l\,|(\mu\nu;\,1)_t\,|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\xi_2\eta_2)=\sigma_{\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)+2^{-\lambda+1}(u_t+w_t)\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\,\xi_2\eta_2)$$
$$+2^{-l-t+1}\Lambda_t\{2^{-\mu}\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\,\xi_1\eta_1)+2^{-\nu}\overline{A}_{\xi_1\eta_1}Q(\alpha\beta;\,\xi_2\eta_2)\}$$
$$+2^{-l-\lambda}(2^{-t+1}\Lambda_t+v_t+2w_t)S(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2),$$
$$\kappa_{l\,(\mu\nu;\,n)_t\,\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sigma_{\mu\nu}(\xi_1\eta_1,\,\xi_2\eta_2)$$
$$+2^{-l-N_t+1}\Lambda_t\{2^{-\mu}\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\,\xi_1\eta_1)+2^{-\nu}\overline{A}_{\xi_1\eta_1}Q(\alpha\beta;\,\xi_2\eta_2)$$
$$+2^{-\lambda}S(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\}\quad\text{for } n_t>1.$$

More generally, we obtain the following results:

$$\kappa_{l\,|(\mu\nu;\,n)_t\,|(\mu'\nu';\,1)_{t'}\,|11}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sigma(\xi_1\eta_1,\,\xi_2\eta_2)+4(u'_{t'}+w'_{t'})\mathfrak{X}(\xi_1\eta_1,\,\xi_2\eta_2)$$
$$+2^{-l-N_t+1}\Lambda_t\{2^{-t'}\Lambda'_{t'}U(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)+(v'_{t'}+2w'_{t'})S(\alpha\theta;\,\xi_1\eta_1,\,\xi_2\eta_2)\},$$
$$\kappa_{l\,(\mu\nu;\,n)_t\,|(\mu'\nu';\,1)_{t'}\,|1\nu'}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)$$
$$=\sigma_{1\nu'}(\xi_1\eta_1,\,\xi_2\eta_2)+2^{-\nu'+1}(u'_{t'}+w'_{t'})\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\,\xi_2\eta_2)$$

$$+2^{-l-N-t'}A_tA'_{t'}\{\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)+2^{-\nu'+1}V(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\}$$
$$+2^{-l-N_t-\nu}A_t(v'_{t'}+2w'_{t'})S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$
$$\kappa_{l\,|\,(\mu\nu;\,n)_t\,|\,(\mu'\nu';\,1)_{t'}\,|\,\mu'\nu'}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$$
$$=\sigma_{\mu'\nu'}(\xi_1\eta_1,\xi_2\eta_2)+2^{-\lambda'+1}(u'_{t'}+w'_{t'})\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2)$$
$$+2^{-l-N_t-t'+1}A_tA'_{t'}\{\bar{2}^{-\mu'}A_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)+\bar{2}^{-\nu'}A_{\xi_1\eta_1}Q(\alpha\beta;\xi_2\eta_2)\}$$
$$+2^{-l-N_t-\lambda'}A_t(2^{-t'+1}A'_{t'}+v'_{t'}+2w'_{t'})S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$

n_t, $\mu'=\mu'_{t'+1}$ and $\nu'=\nu'_{t'+1}$ being supposed to be greater than unity.

It would be noticed that the above formulas except those with $n_t=1$ remain valid even for $l=0$.

4. Descendants combinations after consanguineous marriages

For any mother-descendants combination $(A_{\alpha\beta};A_{\xi_1\eta_1},A_{\xi_2\eta_2})\ldots$, if we eliminate mother's type by summing up the probability $\pi\ldots(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\equiv\bar{A}_{\alpha\beta}\kappa\ldots(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$ over all the possible genotypes of mother, then we obtain the probability of a pair of descendants $(A_{\xi_1\eta_1},A_{\xi_2\eta_2})$ of assigned consanguineous relationship, which will be designated by

$$\sigma\ldots(\xi_1\eta_1,\xi_2\eta_2)=\sum\pi\ldots(ab;\xi_1\eta_1,\xi_2\eta_2)$$
$$\equiv\sum\bar{A}_{\alpha\beta}\kappa\ldots(ab;\xi_1\eta_1,\xi_2\eta_2).$$

In case of a simple mother-descendants combination, $\kappa_{\mu\nu}$, it is given by

$$\sigma_{\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)=\sum\bar{A}_{ab}\kappa_{\mu\nu}(ab;\xi_1\eta_1,\xi_2\eta_2),$$

a case which has been discussed in II, § 1.

We now consider the mother-descendants combination of the form $(A_{\alpha\beta};A_{\xi_1\eta_1},A_{\xi_2\eta_2})_{(\mu\nu;\,1)_t|\mu\nu}$. The probability of the corresponding descendants combination is then given by

$$\sigma_{(\mu\nu;\,1)_t\,11}(\xi_1\eta_1,\xi_2\eta_2)=\sigma(\xi_1\eta_1,\xi_2\eta_2)+4(u_t+w_t)\mathfrak{X}(\xi_1\eta_1,\xi_2\eta_2),$$
$$\sigma_{(\mu\nu;1)_t\,|\,\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)=\sigma_{\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)+2^{-\lambda+1}(u_t+w_t)\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2)\quad\text{for }\mu+\nu>2.$$

These results show that *the distribution of descendants combination* $(A_{\xi_1\eta_1},A_{\xi_2\eta_2})_{(\mu\nu;\,1)_t|\mu\nu}$ *deviates,* compared with one without any consanguineous marrige, by a residual quantity.

$$\sigma_{(\mu\nu;\,1)_t\,|\,\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)-\sigma_{\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)$$
$$=\begin{cases}4(u_t+w_t)\mathfrak{X}(\xi_1\eta_1,\xi_2\eta_2) & \text{for }\mu=\nu=1,\\ (u_t+w_t)\{\sigma_{\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)-\bar{A}_{\xi_1\eta_1}\bar{A}_{\xi_2\eta_2}\} & \text{for }\mu+\nu>2,\end{cases}$$

where we put in conformity with a notation already availed

$$u_t+w_t=\sum_{r=0}^{t-1}\prod_{s=r+1}^{t}2^{-\lambda_s-2}.$$

Whether the deviation occurs in the direction of increase or decrease depends on the sign of values of the factor of u_t+w_t, namely, the sign of values of $\mathfrak{X}(\xi_1\eta_1,\xi_2\eta_2)$ for $\mu=\nu=1$ or of $Q(\xi_1\eta_1;\xi_2\eta_2)$ for $\mu+\nu>2$, respectively.

In case of a mother-descendants combination $(A_{\alpha\beta}; A_{\xi_1\eta_1}, A_{\xi_2\eta_2})_{(\mu\nu; n)_t|\mu\nu}$ with $n_t > 1$, we get

$$\sigma_{(\mu\nu; n)_t|\mu\nu}(\xi_1\eta_1, \xi_2\eta_2) = \sigma_{\mu\nu}(\xi_1\eta_1, \xi_2\eta_2)$$

regardless of the values of μ and ν provided $n_t > 1$, showing that the deviation vanishes out.

In case of a general mother-descendants combination, a similar argument will lead to the formula

$$\sigma_{l|(\mu\nu; n)_t|(\mu'\nu'; 1)_{t'}|\mu'\nu'}(\xi_1\eta_1, \xi_2\eta_2)$$
$$= \begin{cases} \sigma(\xi_1\eta_1, \xi_2\eta_2) + 4(u'_{t'} + w'_{t'})\mathfrak{X}(\xi_1\eta_1, \xi_2\eta_2) & \text{for } \mu' = \nu' = 1, \\ \sigma_{\mu'\nu'}(\xi_1\eta_1, \xi_2\eta_2) + 2^{-\lambda'+1}(u'_{t'} + w'_{t'})\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1; \xi_2\eta_2) & \text{for } \lambda' \equiv \mu' + \nu' - 1 > 1, \end{cases}$$

n_t being supposed to be greater than unity; the formula remains valid even when $l = 0$.

5. Interrelations and asymptotic behaviors of the probabilities

The asymptotic behaviors of the probabilities as one of the generation-numbers tends to infinity can be readily deduced from respective expressions derived above, and will yield several interrelations between the probabilities.

We first observe the behaviors of $\kappa_{l|(\mu\nu; n)_t|\mu\nu}$ as each of μ_r, ν_r and n_r tends to infinity. We obtain the following limit equations:

$$\lim_{\mu\to\infty} \kappa_{l|(\mu\nu; n)_t|\mu\nu}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2) = \bar{A}_{\xi_1\eta_1}\kappa_{l|(\mu\nu; n)_{t-1}|\mu_t\nu_t; n_t+\nu}(\alpha\beta; \xi_2\eta_2),$$

$$\lim_{\mu_z\to\infty} \kappa_{l|(\mu\nu; n)_t|\mu\nu}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2) = \kappa_{l|(\mu\nu; n)_{z-2}|\mu_{z-1}+\nu_z+n_z|(\mu'\nu'; n')_{t-z}|\mu\nu}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2)$$

for $1 \leqq z$ with $\mu'_s = \mu_{z+s}$, $\nu'_s = \nu_{z+s}$, $n'_s = n_{z+s}$ $(1 \leqq s \leqq t - z)$, and

$$\lim_{l\to\infty} \kappa_{l|(\mu\nu; n)_t|\mu\nu}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2) = \sigma_{(\mu\nu; n)_t|\mu\nu}(\xi_1\eta_1, \xi_2\eta_2);$$

here the generation-numbers except one tending to infinity may be quite arbitrary and, in particular, l may be equal to zero, a case which will be easily comprehensible.

An asymptotic behavior of a probability $\kappa_{l|(\mu\nu; n)_t|\mu\nu}$ as t tends to infinity can be deduced similarly as in VI, § 5. We obtain more generally the following results:

If there exists a number τ *such that*

$$\lambda'_r = \lambda'_\infty \text{ (const)} \qquad \textit{for } r > \tau$$

then we have

$$\lim_{t'\to\infty} \kappa_{l|(\mu\nu; n)_t|(\mu'\nu'; 1)_{t'}|\mu'\nu'}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2)$$
$$= \begin{cases} \sigma(\xi_1\eta_1, \xi_2\eta_2) + \dfrac{4}{2^{\lambda'_\infty+2} - 1}\mathfrak{X}(\xi_1\eta_1, \xi_2\eta_2) & \textit{for } \mu' = \nu' = 1, \\ \sigma_{\mu'\nu'}(\xi_1\eta_1, \xi_2\eta_2) + \dfrac{2^{-\mu'-\nu'}}{2^{\lambda'_\infty+2} - 1}\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1; \xi_2\eta_2) & \textit{for } \mu' + \nu' > 2, \end{cases}$$

and if

$$\lim_{r\to\infty} \lambda'_r = \infty$$

then we have

$$\lim_{t'\to\infty} \kappa_{l|(\mu\nu; n)_t|(\mu'\nu'; 1)_{t'}|\mu'\nu'}(\alpha\beta; \xi_1\eta_1, \xi_2\eta_2) = \sigma_{\mu'\nu'}(\xi_1\eta_1, \xi_2\eta_2);$$

otherwise, the probability under consideration will oscillate, as $t' \to \infty$, *within certain upper and lower bounds.*