51. *Probabilities on Inheritance in Consanguineous Families. VII*

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VII. Mother-descendants combinations through several consanguineous marriages

1. Special combinations with several consanguineous marriages

The main purpose of the present chapter is to determine the probability of a mother-descendants combination designated by

$$\pi_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\equiv\bar{A}_{\alpha\beta}\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\qquad(\mu=\mu_{t+1},\ \nu=\nu_{t+1}).$$

By definition, the reduced probability $\kappa_{(\mu\nu;n)_t|\mu\nu}$ is given by

$$\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;n)_t}(\alpha\beta;\,ab)\kappa_{\mu\nu}(ab;\,\xi_1\eta_1,\,\xi_2\eta_2).$$

Evidently, this probability is symmetric with respect to μ_r and ν_r for any r with $1\leqq r\leqq t$, while it is quasi-symmetric with respect to μ and ν, i.e.

$$\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\kappa_{(\mu\nu;n)_t|\nu\mu}(\alpha\beta;\,\xi_2\eta_2,\,\xi_1\eta_1).$$

In the present section we first deal with the case where the n_r, μ and ν are all equal to unity. After substituting the known expressions, its defining equation yields

$$\begin{aligned}\kappa_{(\mu\nu;1)_t|11}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)&=\sigma(\xi_1\eta_1,\,\xi_2\eta_2)+2^{-t+1}\varLambda_tU(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)\\&\quad+4u_t\sum R(ab)\kappa(ab;\,\xi_1\eta_1,\,\xi_2\eta_2)+2v_t\sum S(\alpha\beta;\,ab)\kappa(ab;\,\xi_1\eta_1,\,\xi_2\eta_2)\\&\quad+4w_t\sum T(\alpha\beta;\,ab)\kappa(ab;\,\xi_1\eta_1,\,\xi_2\eta_2).\end{aligned}$$

Thus, it remains only to determine the last three residual terms, i.e.

$$\mathfrak{X}(\xi_1\eta_1,\,\xi_2\eta_2)=\sum R(ab)\kappa(ab;\,\xi_1\eta_1,\,\xi_2\eta_2),$$
$$\mathfrak{Y}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sum S(\alpha\beta;\,ab)\kappa(ab;\,\xi_1\eta_1,\,\xi_2\eta_2),$$
$$\mathfrak{Z}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)=\sum T(\alpha\beta;\,ab)\kappa(ab;\,\xi_1\eta_1,\,\xi_2\eta_2),$$

which are evidently symmetric with respect to $\xi_1\eta_1$ and $\xi_2\eta_2$. Actual computation leads to the following results:

$$\mathfrak{X}(ii,\,ii)=\tfrac{1}{8}i^2(1-i)(1+i),\qquad \mathfrak{X}(ii,\,ik)=-\tfrac{1}{4}i^3k,$$
$$\mathfrak{X}(ii,\,kk)=-\tfrac{1}{8}i^2k^2,\qquad \mathfrak{X}(ii,\,hk)=-\tfrac{1}{4}i^2hk,$$
$$\mathfrak{X}(ij,\,ij)=\tfrac{1}{4}ij(1-2ij),\qquad \mathfrak{X}(ij,\,ik)=-\tfrac{1}{2}i^2jk,$$
$$\mathfrak{X}(ij,\,hk)=-\tfrac{1}{2}ijhk;$$
$$\mathfrak{Y}(ii;\,ii,\,ii)=-\tfrac{1}{16}i(1-i)^2(1-2i),\qquad \mathfrak{Y}(ii;\,ii,\,ig)=\tfrac{1}{8}ig(1-i)(1-2i),$$
$$\mathfrak{Y}(ii;\,ii,\,gg)=-\tfrac{1}{16}ig^2(1-2i),\qquad \mathfrak{Y}(ii;\,ii,fg)=-\tfrac{1}{8}ifg(1-2i),$$
$$\mathfrak{Y}(ii;\,ik,\,ik)=-\tfrac{1}{16}k(1-4i-k+3i^2-8i^2k),$$
$$\mathfrak{Y}(ii;\,ik,\,kk)=\tfrac{1}{16}k^2(1+i-5k+4ik),$$
$$\mathfrak{Y}(ii;\,ik,\,ig)=\tfrac{1}{16}kg(1-7i+8i^2),\qquad \mathfrak{Y}(ii;\,ik,\,kg)=\tfrac{1}{16}kg(1-3i-2k+8ik),$$
$$\mathfrak{Y}(ii;\,ik,\,gg)=-\tfrac{1}{16}kg^2(1-4i),\qquad \mathfrak{Y}(ii;\,ik,fg)=-\tfrac{1}{8}kfg(1-4i),$$

$\mathfrak{Y}(ii;\,kk,\,kk)=\frac{1}{16}k^2(1-k)(1-2k),\qquad \mathfrak{Y}(ii;\,kk,\,kg)=-\frac{1}{16}k^2g(3-4k),$
$\mathfrak{Y}(ii;\,kk,\,gg)=\frac{1}{8}k^2g^2,\qquad \mathfrak{Y}(ii;\,kk,\,fg)=\frac{1}{4}k^2fg,$
$\mathfrak{Y}(ii;\,hk,\,hk)=\frac{1}{16}hk(2-3h-3k+8hk),\qquad \mathfrak{Y}(ii;\,hk,\,kg)=-\frac{1}{16}hkg(3-8k),$
$\mathfrak{Y}(ii;\,hk,\,fg)=\frac{1}{2}hkfg,$
$\mathfrak{Y}(ij;\,ii,\,ii)=-\frac{1}{32}i(1-i)(1+i^2),\qquad \mathfrak{Y}(ij;ii,ij)=\frac{1}{32}i(i+2j-i^2-5ij+8i^2j),$
$\mathfrak{Y}(ij;\,ii,\,jj)=-\frac{1}{32}ij(i+j-4ij),\qquad \mathfrak{Y}(ij;\,ii,\,ig)=\frac{1}{32}ig(2-5i+8i^2),$
$\mathfrak{Y}(ij;\,ii,\,jg)=-\frac{1}{32}ig(i+2j-8ij),\qquad \mathfrak{Y}(ij;\,ii,\,gg)=-\frac{1}{32}ig^2(1-4i),$
$\mathfrak{Y}(ij;\,ii,\,fg)=-\frac{1}{16}ifg(1-4i),$
$\mathfrak{Y}(ij;\,ij,\,ij)=-\frac{1}{32}(i+j-i^2-j^2+6ij(i+j)-16i^2j^2),$
$\mathfrak{Y}(ij;\,ij,\,ig)=\frac{1}{32}g(i+j-2i^2-6ij+16i^2j),$
$\mathfrak{Y}(ij;\,ij,\,gg)=-\frac{1}{32}g^2(i+j-8ij),\qquad \mathfrak{Y}(ij;\,ij,\,fg)=-\frac{1}{16}fg(i+j-8ij),$
$\mathfrak{Y}(ij;\,ik,\,ik)=-\frac{1}{32}k(1+2i-k+4i^2+6ik-16i^2k),$
$\mathfrak{Y}(ij;\,ik,\,jk)=\frac{1}{32}k(i+j-6ij-2(i+j)k+16ijk),$
$\mathfrak{Y}(ij;\,ik,\,kk)=\frac{1}{32}k^2(1-6i-k+8ik),\qquad \mathfrak{Y}(ij;\,ik,\,ig)=\frac{1}{32}kg(1-2i)(1-8i),$
$\mathfrak{Y}(ij;\,ik,\,jg)=-\frac{1}{16}kg(i+j-8ij),\qquad \mathfrak{Y}(ij;\,ik,\,kg)=\frac{1}{32}kg(1-6i-2k+16ik),$
$\mathfrak{Y}(ij;\,ik,\,gg)=-\frac{1}{32}kg^2(1-8i),\qquad \mathfrak{Y}(ij;\,ik,\,fg)=-\frac{1}{16}kfg(1-8i);$
$\mathfrak{Z}(ii;\,ii,\,ii)=\frac{1}{16}i(1-i)(2-i)(1+i),\qquad \mathfrak{Z}(ii;\,ii,\,ig)=-\frac{1}{8}i^2g(2-i),$
$\mathfrak{Z}(ii;\,ii,\,gg)=-\frac{1}{16}ig^2(2-i),\qquad \mathfrak{Z}(ii;\,ii,\,fg)=-\frac{1}{8}ifg(2-i),$
$\mathfrak{Z}(ii;\,ik,\,ik)=\frac{1}{8}k(1-i+i^2+k(1-i)(1-2i)),$
$\mathfrak{Z}(ii;\,ik,\,kk)=-\frac{1}{8}k^2(1+k)(1-i),$
$\mathfrak{Z}(ii;\,ik,\,ig)=\frac{1}{8}kg(1-i)(1-2i),\qquad \mathfrak{Z}(ii;\,ik,\,kg)=-\frac{1}{8}kg(1+2k)(1-i),$
$\mathfrak{Z}(ii;\,ik,\,gg)=-\frac{1}{8}kg^2(1-i),\qquad \mathfrak{Z}(ii;\,ik,fg)=-\frac{1}{4}kfg(1-i),$
$\mathfrak{Z}(ii;\,kk,\,kk)=\frac{1}{16}k^2(1+k)^2,\qquad \mathfrak{Z}(ii;\,kk,\,kg)=\frac{1}{8}k^2g(1+k),$
$\mathfrak{Z}(ii;\,kk,\,gg)=\frac{1}{16}k^2g^2,\qquad \mathfrak{Z}(ii;\,kk,\,fg)=\frac{1}{8}k^2fg,$
$\mathfrak{Z}(ii;\,hk,\,hk)=\frac{1}{8}hk(1+h+k+2hk),\qquad \mathfrak{Z}(ii;\,hk,\,kg)=\frac{1}{8}hkg(1+2k),$
$\mathfrak{Z}(ii;\,hk,\,fg)=\frac{1}{4}hkfg,$
$\mathfrak{Z}(ij;\,ii,\,ii)=\frac{1}{32}i(1+i)(1-2i+2i^2),$
$\mathfrak{Z}(ij;\,ii,\,ij)=\frac{1}{32}i(1-i-j-2i^2-2ij+4i^2j),$
$\mathfrak{Z}(ij;\,ii,\,jj)=\frac{1}{32}ij(1-2i-2j+2ij),\qquad \mathfrak{Z}(ij;\,ii,\,ig)=-\frac{1}{32}ig(1+2i-4i^2),$
$\mathfrak{Z}(ij;\,ii,\,jg)=\frac{1}{32}ig(1-2i-4j+4ij),\qquad \mathfrak{Z}(ij;\,ii,\,gg)=-\frac{1}{16}ig^2(1-i),$
$\mathfrak{Z}(ij;\,ii,fg)=-\frac{1}{8}ifg(1-i),$
$\mathfrak{Z}(ij;\,ij,\,ij)=\frac{1}{32}((1-4ij)(i+j-2ij)+i^2+j^2),$
$\mathfrak{Z}(ij;\,ij,\,ig)=-\frac{1}{32}g(i-j+4i^2+4ij-8i^2j),$
$\mathfrak{Z}(ij;\,ij,\,gg)=-\frac{1}{16}g^2(i+j-2ij),\qquad \mathfrak{Z}(ij;\,ij,\,fg)=-\frac{1}{8}fg(i+j-2ij),$
$\mathfrak{Z}(ij;\,ik,\,ik)=\frac{1}{32}k(1+k+4i^2-4ik+8i^2k),$
$\mathfrak{Z}(ij;\,ik,\,jk)=\frac{1}{32}k(1-2(i+j)+k+4ij-4(i+j)k+8ijk),$
$\mathfrak{Z}(ij;\,ik,\,kk)=-\frac{1}{16}k^2(1+k)(1-2i),\qquad \mathfrak{Z}(ij;\,ik,\,ig)=\frac{1}{32}kg(1-4i+8i^2),$
$\mathfrak{Z}(ij;\,ik,\,jg)=\frac{1}{32}kg(1-4i-4j+8ij),\qquad \mathfrak{Z}(ij;ik,kg)=-\frac{1}{16}kg(1+2k)(1-2i),$
$\mathfrak{Z}(ij;\,ik,\,gg)=-\frac{1}{16}kg^2(1-2i),\qquad \mathfrak{Z}(ij;\,ik,\,fg)=-\frac{1}{8}kfg(1-2i).$

Since, besides an evident symmetry character of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ with respect to descendants' types, the quantities $\mathfrak{Y}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)$ and $\mathfrak{Z}(\alpha\beta;\,\xi_1\eta_1,\,\xi_2\eta_2)$ are both independent of $\alpha\beta$ provided $A_{\xi_1\eta_1}$ and $A_{\xi_2\eta_2}$ have no gene in common with $A_{\alpha\beta}$, all the possible cases have thus been essentially worked out. The final expression for desired probability is thus written in the form

$$\kappa_{(\mu\nu;1)_t|11}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma(\xi_1\eta_1,\xi_2\eta_2)+2^{-t+1}U(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$$
$$+4u_t\mathfrak{X}(\xi_1\eta_1,\xi_2\eta_2)+2v_t\mathfrak{Y}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)+4w_t\mathfrak{Z}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$

which remains true also for $t=1$ with $u_1=v_1=0$.

By the way, we can readily deduce the relations

$$\sum\mathfrak{X}(\xi\eta,ab)=\sum\mathfrak{Y}(\alpha\beta;\xi\eta,ab)=\sum\mathfrak{Z}(\alpha\beta;\xi\eta,ab)=0.$$

We further remark that there hold the relations

$$\sum\bar{A}_{ab}\mathfrak{Y}(ab;\xi_1\eta_1,\xi_2\eta_2)=0,\quad \sum\bar{A}_{ab}\mathfrak{Z}(ab;\xi_1\eta_1,\xi_2\eta_2)=\mathfrak{X}(\xi_1\eta_1,\xi_2\eta_2),$$
$$\sum Q(\alpha\beta;ab)\mathfrak{Y}(ab;\xi_1\eta_1,\xi_2\eta_2)=\sum Q(\alpha\beta;ab)\mathfrak{Z}(ab;\xi_1\eta_1,\xi_2\eta_2)=\tfrac{1}{2}\mathfrak{Y}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2).$$

We now consider the probability $\kappa_{(\mu\nu;n)_t|11}$ in which $n_t>1$ while $n_r\geqq 1$ for $1\leqq r<t$. It is defined by the equation

$$\kappa_{(\mu\nu;n)_t|11}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;n)_t}(\alpha\beta;ab)\kappa(ab;\xi_1\eta_1,\xi_2\eta_2).$$

Hence, we get the formula

$$\kappa_{(\mu\nu;n)_t|11}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma(\xi_1\eta_1,\xi_2\eta_2)+2^{-N_t+1}\varLambda_tU(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$

which remains true even when $t=1$.

We next observe the probability $\kappa_{(\mu\nu;1)_t|1\nu}$ with $\nu>1$. The defining equation then yields

$$\kappa_{(\mu\nu;1)_t|1\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;1)_t}(\alpha\beta;ab)\kappa_{1\nu}(ab;\xi_1\eta_1,\xi_2\eta_2).$$

Based on the relation

$$\sum R(ab)W(ab;\xi_1\eta_1,\xi_2\eta_2)=\tfrac{1}{2}\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2),$$
$$\sum S(\alpha\beta;ab)W(ab;\xi_1\eta_1,\xi_2\eta_2)=\tfrac{1}{2}S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$
$$\sum T(\alpha\beta;ab)W(ab;\xi_1\eta_1,\xi_2\eta_2)=\tfrac{1}{4}T(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$

we get the formula in the form

$$\kappa_{(\mu\nu;1)_t|1\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma_{1\nu}(\xi_1\eta_1,\xi_2\eta_2)$$
$$+2^{-t}\varLambda_t\{\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)+2^{-\nu+1}V(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\}$$
$$+2^{-\nu}\{2u_t\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2)+v_tS(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)+w_tT(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\},$$

which remains valid also for $t=1$ with $u_1=v_1=0$.

We finally consider the probability $\kappa_{(\mu\nu;n)_t|1\nu}$ in which n_t and ν are greater than unity while the n_r's with $1\leqq r<t$ may be arbitrary numbers equal to or greater than unity. Its defining equation becomes

$$\kappa_{(\mu\nu;n)_t|1\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;n)_t}(\alpha\beta;ab)\kappa_{1\nu}(ab;\xi_1\eta_1,\xi_2\eta_2),$$

and hence readily leads to the desired formula

$$\kappa_{(\mu\nu;n)_t|1\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma_{1\nu}(\xi_1\eta_1,\xi_2\eta_2)$$
$$+2^{-N_t}\varLambda_t\{\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)+2^{-\nu+1}V(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\},$$

which remains true even when t is equal to unity.

2. Generic combinations with several consanguineous marriages

We consider the probability $\kappa_{(\mu\nu;n)_t|\mu\nu}$ for the generic case, namely with $\mu,\nu>1$.

We first deal with the case where the n's are all equal to unity.

It is defined by the equation

$$\kappa_{(\mu\nu;1)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;1)_t}(\alpha\beta;ab)\kappa_{\mu\nu}(ab;\xi_1\eta_1,\xi_2\eta_2).$$

Based on the identical relations

$$\sum R(ab)T(ab;\xi_1\eta_1,\xi_2\eta_2)=\tfrac{1}{2}\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2),$$
$$\sum S(\alpha\beta;ab)T(ab;\xi_1\eta_1,\xi_2\eta_2)=\tfrac{1}{2}S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$
$$\sum T(\alpha\beta;ab)T(ab;\xi_1\eta_1,\xi_2\eta_2)=\tfrac{1}{4}T(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2),$$

the desired formula is then given, with $\lambda=\mu+\nu-1$, by

$$\begin{aligned}\kappa_{(\mu\nu;1)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)&=\sigma_{\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)+2^{-\lambda+1}u_t\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2)\\&+2^{-t+1}\Lambda_t\{2^{-\mu}\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)+2^{-\nu}\overline{A}_{\xi_1\eta_1}Q(\alpha\beta;\xi_2\eta_2)\}\\&+2^{-\lambda}\{(2^{-t+1}\Lambda_t+v_t)S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)+w_tT(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\}.\end{aligned}$$

We next observe the most generic case, i.e. $\kappa_{(\mu\nu;n)_t|\mu\nu}$ with n_t, μ, $\nu>1$. Its defining equation becomes

$$\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;n)_t}(\alpha\beta;ab)\kappa_{\mu\nu}(ab;\xi_1\eta_1,\xi_2\eta_2)$$

and it leads to the desired formula

$$\begin{aligned}&\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma_{\mu\nu}(\xi_1\eta_1,\xi_2\eta_2)\\&\quad+2^{-N_t+1}\Lambda_t\{2^{-\mu}\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)+2^{-\nu}\overline{A}_{\xi_1\eta_1}Q(\alpha\beta;\xi_2\eta_2)+2^{-\lambda}S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\},\end{aligned}$$

which remains valid even when $t=1$.

Finally, we consider the probability $\kappa_{(\mu\nu;n)_t|(\mu'\nu';1)_{t'}|\mu'\nu'}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$ $(\mu'\nu'=\mu'_{t'+1}\nu'_{t'+1})$. The defining equation is then given by

$$\begin{aligned}&\kappa_{(\mu\nu;n)_t|(\mu'\nu';1)_{t'}|\mu'\nu'}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\\&\quad=\sum\kappa_{(\mu\nu;n)_t|(\mu'\nu';1)_{t'}}(\alpha\beta;ab)\kappa_{\mu'\nu'}(ab;\xi_1\eta_1,\xi_2\eta_2).\end{aligned}$$

We distinguish here three systems according to the generation-numbers (μ',ν') of the descendants $(A_{\xi_1\eta_1},A_{\xi_2\eta_2})$: $\mu'=\nu'=1$; $\mu'=1<\nu'$ or $\mu'>1=\nu'$; $\mu',\nu'>1$. It is shown that there hold the following formulas:

$$\begin{aligned}&\kappa_{(\mu\nu;n)_t|(\mu'\nu';1)_{t'}|11}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma(\xi_1\eta_1,\xi_2\eta_2)+4(u'_{t'}+w'_{t'})\mathfrak{X}(\xi_1\eta_1,\xi_2\eta_2)\\&\qquad+2^{-N_t+1}\Lambda_t\{2^{-t'}\Lambda'_{t'}U(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)+(v'_{t'}+2w'_{t'})\mathfrak{Y}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\},\\&\kappa_{(\mu\nu;n)_t|(\mu'\nu';1)_{t'}|1\nu'}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma_{1\nu'}(\xi_1\eta_1,\xi_2\eta_2)\\&\qquad+2^{-\nu'+1}(u'_{t'}+w'_{t'})\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2)+2^{-N_t}\Lambda_t\{2^{-t'}\Lambda'_{t'}\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)\\&\qquad+2^{-\nu'-t'+1}\Lambda'_{t'}V(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)+2^{-\nu'}(v'_{t'}+2w'_{t'})S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\}\\&\kappa_{(\mu\nu;n)_t|(\mu'\nu';1)_{t'}|\mu'\nu'}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sigma_{\mu'\nu'}(\xi_1\eta_1,\xi_2\eta_2)\\&\qquad+2^{-\lambda'+1}(u'_{t'}+w'_{t'})\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\xi_2\eta_2)\\&\qquad+2^{-N_t-t'+1}\Lambda_t\Lambda'_{t'}\{2^{-\mu'}\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)+2^{-\nu'}\overline{A}_{\xi_1\eta_1}Q(\alpha\beta;\xi_2\eta_2)\}\\&\qquad+2^{-N_t-\lambda'}\Lambda_t(2^{-t'+1}\Lambda'_{t'}+v'_{t'}+2w'_{t'})S(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\end{aligned}$$

with $\lambda'=\mu'+\nu'-1$.