35. On Symbolic Representation

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M. Morse and G. A. Hedlund have shown the method of symbolic dynamics and proved the interesting theorems in their papers [1], [2]. Those theorems seem as if they are independent of classical dynamics but they are indeed a new representation of classical interesting theorems

In this paper we shall prove the theorems of symbolic representation. These theorems are applicable to transitivity problem.

1. We consider a closed two-dimensional Riemannian manifold $\sum$ which is of genus $p \geqq 1$. The adding assumption is that no geodesic on $\sum$ has on it two mutually conjugate points.

When $p>1$ a convex domain S_0 in the unit circle regarded as the non-Euclidean plane Φ is bounded by a sequence

$$B_1^{-1}, A_1^{-1}, B_1, A_1, B_2^{-1}, A_2^{-1}, B_2, A_2, \dots, B_p^{-1}, A_p^{-1}, B_p, A_p,$$

of congruent segments of H-straight (H means hyperbolic) lines such that each pair of the successive H-lines forms an angle equal to $\frac{\pi}{2p}$. If we identify congruent points of conjugate sides of S_0, we get a closed orientable surface T of genus p with constant negative curvature.

Let $\tilde{B}_1, \tilde{A}_1, \tilde{B}_2, \tilde{A}_2, \dots, \tilde{B}_p, \tilde{A}_p$ be a set of geodesics which starts from and comes back to a point P of $\sum$ and every geodesic of the set be. homotopic to a curve of canonical section of $\sum$. Then we can select those geodesics so as to be independent each other if we choose P suitably.

We map $\sum$ topologically on T and $\tilde{A}_i, \tilde{B}_i$ on A_i, B_i respectively and denote this map f.

When $p=1$ a convex domain S_0 in Euclidean plane Φ is bounded by a sequence

$$B_1^{-1}, A_1^{-1}, B_1, A_1,$$

of congruent segments of E-straight lines (E means Euclidean) such that each pair of successive H-lines forms an angle equal to $\frac{\pi}{2}$ If we identify congruent points of conjugate sides of S_0, we get a closed orientable surface T of genus 1 with vanishing curvature.

Let $\tilde{B}_1, \tilde{A}_1$ be geodesics which start from and come back to a point P of $\sum$ and each geodesic be homotopic to a curve of a canonical section of $\sum$ Then we can select those geodesics so as to be independent each other if we choose P suitably We map $\sum$ topologically on T and $\tilde{A}_1, \tilde{B}_1$ on A_1, B_1 respectively and denote this map f.

In general we consider the H-straight segments which pass from an interior point of S_0 to an interior point of $B_i^{-1}, A_i^{-1}, B_i, A_i$ $(1=1, 2, 3, \dots, p)$ and we denote those $a_i, b_i^{-1}, a_i^{-1}, b_i$ respectively. Then $f^{-1}(a_i), f^{-1}(b_i^{-1}), f^{-1}(a_i^{-1}), f^{-1}(b_i)$ will be denoted as $\tilde{a}_i, \tilde{b}_i^{-1}, \tilde{a}_i^{-1}, \tilde{b}_i$ respectively.

We prepare some definitions.

Definition 1.1. We shall call the curves on T which are images of geodesics on $\sum$ as geodesics on T.

Definition 1.2. A geodesic segment h will be said to be of class A if h on $\sum$ is at least as short as any other rectifiable curve joining h's end points and capable of being continuously deformed on $\sum$ into h, without moving its end points. The image of geodesic which is of class A on $\sum$ by f is called as geodesic of class A.

An unending geodesic (on $\sum$ or T) will be said to be of class A if each of its finite segments is of class A.

Definition 1.3. Two unending curves on Φ will be said to be of the same type if there exists positive constant C such that every point of either curve lies in H-distance less than C from some point of the other.

Two unending curves on $\sum$ will be said to be of the same type if there exists at least a pair of the same types in those images by f.

Definition 1.4. We shall say that $\sum$ satisfies the hypothesis of unicity if there is only one geodesic of class A whose image on S is of the type of each H-straight line when $p>1$ or E-straight line when $p=1$.

Definition 1.5. Geodesic on $\sum$ will be said to be regular relative to P when it does not pass a common point P of $\tilde{B}_i$ and $\tilde{A}_i$. Geodesic on $\sum$ which contains $\tilde{A}_i$ or $\tilde{B}_i$ for some i will be called to be special geodesic relative to P. The image of regular, special geodesic relative to P by f will be called also regular, special geodesic.

2. Let $\theta_1, \theta_2, \dots, \theta_{4p}$ be the angles made by the successive two of $\tilde{B}_1^{-1}, \tilde{A}_1^{-1}, \tilde{B}_1, \tilde{A}_1, \dots, \tilde{B}_p^{-1}, \tilde{A}_p^{-1}, \tilde{B}_p, \tilde{A}_p, \tilde{B}_1^{-1}$, respectively. These angles arrange around P on $\sum$ in order of $\theta_3, \theta_2, \theta_1, \theta_4, \theta_7, \theta_6, \theta_5, \theta_8, \dots, \theta_{4p-1}, \theta_{4p-2}, \theta_{4p-3}, \theta_{4p}, \theta_3, \theta_2, \dots$ and their sums of the successive $4p$ angles are 2π. We assume that the sum from the i-th angle

to the $(i+r_i-3)$-th angle is not greater than π and the sum from the i-th angle to the $(i+r_i-2)$-th angle is greater than π. r_i is constant depending upon P and i only.

Definition 2.1. A symbolic sequence (cf. Morse-Hedlund [1], [2]) will be called regular when it satisfies the following conditions:

1) No element (generator) immediately follows its inverse.
2) Subblock of length r_i or greater than r_i beginning from i-th symbol of

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_pb_pa_p^{-1}b_p^{-1}a_1b_1\dots$$

does not exist.

The proof of the following theorem depends on the proof of Morse's theorem. (Cf. Morse [1].)

Theorem 1. If there be given any regular geodesic relative to P on $\sum$, there exists one, and only one unending regular sequence whose generating symbols are $\tilde{a}_i, \tilde{b}_i, \tilde{a}_i^{-1}, \tilde{b}_i^{-1}$.

Proof. Let g be a regular geodesic relative to P on $\sum$. Then $f(g)$ is regular on Φ, and $f(g)$ does not pass $f(P)$. Therefore $f(g)$ crosses the interior point of sides of S_0. As Φ is a universal covering space of S_0, $f(g)$ crosses the interior point of the sides which are images of the sides of S_0 by the transformation of G, where G is Fuchsian group for $p>1$, and analogous group for $p=1$. (Cf. Ford [1].)

Hence $f(g)$ is corresponded by an unending symbolic sequence whose generators are $b_i, a_i, b_i^{-1}, a_i^{-1}$. So g is corresponded by an analogous sequence whose generators are $\tilde{b}_i, \tilde{a}_i, \tilde{b}_i^{-1}, \tilde{a}_i^{-1}$. This correspondence is induced by map f.

As $\sum$ satisfies the condition that no geodesic on $\sum$ has on it two mutually conjugate points, g is of class A. (Cf. Morse-Hedlund [3].) Then $f(g)$ is of class A and it does not happen to enter into the image of S_0 from one side and immediately goes out from the same one. As f is topological, g satisfies the similar condition. Hence a symbol and its successor are not inverse in symbolic sequence corresponding to g. As g is of class A, any segment $f(g)$ lies on convex domain of Φ, where convex is used by the method of geodesic of Definition 1.1. Therefore there is not a subblock from the i-th symbol to the $(i+r_i-1)$-th symbol of

$$\tilde{a}_1\tilde{b}_1\tilde{a}_1^{-1}\tilde{b}_1^{-1}\dots\tilde{a}_p\tilde{b}_p\tilde{a}_p^{-1}\tilde{b}_p^{-1}\tilde{a}_1\tilde{b}_1\dots,$$

because of the assumption of the interior angles of $f^{-1}(S_0)$. Then the second condition of the regular sequence is satisfied.

Theorem 2. If there be given any unending regular sequence whose generating symbols are $\tilde{a}_i, \tilde{b}_i, \tilde{a}_i^{-1}, \tilde{b}_i^{-1}$, there exists at least one geodesic which corresponds to the given regular sequence.

Proof. An unending regular sequence

$$\ldots h_{-2}h_{-1}h_0h_1h_2\ldots$$

having the generating symbols $\tilde{a}_i, \tilde{b}_i, \tilde{a}_i^{-1}, \tilde{b}_i^{-1}$, determines an unending linear set L of the image of S_0. As the first condition of the regular sequence is satisfied, the image of S_0 is not used twice. If we take care of the second condition of the regular sequence and the conditions of interior angles of $f^{-1}(S_0)$, L determines the convex domain (by the method of geodesic of Definition 1.1). When L is represented by

$$(1) \qquad \ldots S_{-2}S_{-1}S_0S_1S_2\ldots$$

we denote g_n the geodesic segment which passes from an interior point of S_{-n} to an interior point of S_n lying on (1).

It is evident that there is such a geodesic segment. We denote by e_n a line element which lies on S_0 and g_n. The set of e_n has a limit element e. I will show that geodesic g^* determined by e has a point on each S_i of (1). Let r be any positive integer. For an integer $n > r$ the portion of g_n in

$$(2) \qquad S_{-r}S_{-r+1}\ldots S_{-2}S_{-1}S_0S_1S_2\ldots S_{r-1}S_r$$

is less in length than some fixed quantity independent of n. A finite segment of a geodesic varies continuously with its initial element. It follows that g^* possesses a finite segment g_r^* which has a point in each S_i of (2) and is wholly contained in (2). From the fact that g_r^* has a point in each S_i in (2), we may conclude that g^* has a point in each S_i in (1). For r sufficiently large, any given segment of g^* that begins with a point of S_0, is included in one of the two portions into which g_r^* is divided by those points. Thus every point of g^* lies on some segment g_r^*. Every point of g_r^* and hence every point of g^* lies in the given linear set. g^* is not special and does not pass $f(P)$. Then g^* is a regular geodesic
Now I assume that a geodesic on $\sum$ satisfies the condition of uniform instability. (Cf. Morse [2].) Then we know the following Morse's theorem.

Theorem 3. If a geodesic on $\sum$ satisfies the condition of uniform instability, it satisfies the hypothesis of unicity. When we consider theorems 1 and 2, we take care of the assumption and the conclusion of theorem 3. Then we can prove the following theorem easily, whose genus is $p>1$.

Theorem 4. If a geodesic on $\sum$ satisfies the condition of uniform instability, there is one-to-one correspondence between the set of all regular geodesics relative to some fixed point P on $\sum$

and the set of all regular unending sequences whose generating symbols are $\tilde{a}_i, \tilde{b}_i, \tilde{a}_i^{-1}, \tilde{b}_i^{-1}$.

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