62. *Shoda's Condition on Quasi-Frobenius Rings*

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Let A be a ring with minimum condition for left and right ideals. Then the following conditions C-1, ..., C-4 are equivalent which has been proved by Nakayama and Ikeda. In this note we shall give somewhat simpler proofs of Propositions 2 and 3 of Ikeda [2], directly deducing from Shoda's condition (to be explained below) the *annihilator relations* in A.

As is well known, we have two direct decompositions of A:

$$A=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}+l(E)=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}A+r(E)$$

where $E=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}$, and $e_{\kappa,i}$ ($\kappa=1, 2, \ldots, k$; $i=1, 2, \ldots, f(\kappa)$) are mutually orthogonal primitive idempotents, and $l(*)$, ($r(*)$) is the left (right) annihilator of $*$. For the sake of brevity, let us denote one of $e_{\kappa,i}$ ($i=1, 2, \ldots, f(\kappa)$), say $e_{\kappa,1}$, by e_κ.

C-1) A possesses a unit element and there exists a permutation $(\pi(1), \ldots, \pi(k))$ of $(1, 2, \ldots, k)$ such that for each κ,

i) $e_\kappa A$ has a unique simple right subideal $\mathfrak{r}_\kappa$ and $\mathfrak{r}_\kappa \cong \bar{e}_{\pi(\kappa)}\bar{A}$,

ii) $Ae_{\pi(\kappa)}$ has a unique simple left subideal $\mathfrak{l}_{\pi(\kappa)}$ and $\mathfrak{l}_{\pi(\kappa)} \cong \bar{A}\bar{e}_\kappa$.

(This is the definition of quasi-Frobenius rings.)

C-2) Annihilator relations $\mathfrak{l}(r(\mathfrak{l}))=\mathfrak{l}$, and $r(l(\mathfrak{r}))=\mathfrak{r}$ hold for every left ideal $\mathfrak{l}$ and every right ideal $\mathfrak{r}$.

C-3) A satisfies Shoda's condition for simple left and right ideals.

C-4) A has a left unit element and satisfies Shoda's condition for every left ideal.

C-4′) A has a left unit and every homomorphism between a left ideal and a simple left ideal is given by the right multiplication of an element of A.

We understand the following as Shoda's condition: Every homorphism between two left (right) ideals is given by the right (left) multiplication of an element of A. We require an auxiliary result: If A satisfies Shoda's condition for simple left ideals, then A has a right unit element. This is Lemma 1 in [2].

Lemma. If we assume Shoda's condition for every simple left ideal, and A has a left unit element and Aa is a simple left ideal, *then aA is a simple right ideal, and $r(N)\subseteqq l(N)$.*

Proof: Let x be an element of A, such that $ax\neq 0$. Then $Aa\cong Aax$. Thus by Shoda's condition, there exists an element x' such that, $a=axx'$ thus $axA\ni axx'=a$, i.e., $axA\supseteqq aA$. Thus aA is a simple right ideal.

Let a be a non-zero element of a simple left ideal, then Aa is a simple left ideal. Thus aA is a simple right ideal, and since we have a right unit by Lemma 1, in [2], $a\in aA$, i.e., $a\in l(N)$. Thus $r(N)\subseteqq l(N)$.

Theorem. Conditions 1, 2, 3, and 4′ are equivalent to each other.

Proof:

1) C-4′) $\to$ C-3).

i) There exists an element a in A such that $\bar{A}\bar{e}_\kappa\cong Aa$ for every κ.

For, if Aae_κ and Abe_λ are simple left ideals and isomorphic for $\kappa\neq\lambda$, then by Shoda's condition $Aae_\kappa=Abe_\lambda xe_\kappa$ for suitable $x\in A$, and then $Abe_\lambda xe_\kappa\subseteqq r(N)E_\kappa$ and since $r(N)E_\lambda$ is two-sided, (since $r(N)\subseteqq l(N)$ by our Lemma, $r(N)E_\lambda A=r(N)E_\lambda(\sum_\tau E_\tau AE_\tau\smallsmile N)=r(N)E_\lambda$) the left hand side is in $r(N)E_\lambda$, that is $Aae_\kappa=Abe_\lambda xe_\kappa=0$.

ii) Let aA be a simple right ideal, then Aa is a simple left ideal.

For, let Q (left ideal) be maximal in Aa, then $Aa/Q\cong\bar{A}\bar{e}_\kappa\cong Aa'$ for suitable e_κ and a'. Thus by Shoda's condition, $Aa'=Aaa''$ for suitable $a''\in A$, and since $aa''A=aA$, there exists an element $x\in A$, and $aa''x=a$. Now since Aaa'' is a simple left ideal, $Aaa''x=Aa$ is so.

iii) If two simple right ideals aA and bA are isomorphic, and a corresponds to b by this isomorphism, then $Aa=Ab$.

For, if $Aa\neq Ab$, then $Aa\frown Ab=0$, then $Aa+Ab/Ab\cong Aa$. Thus by Shoda's condition, this homorphism, between $Aa+Ab$ and Aa is given by right multiplication of an element x of A. And then $Abx=0$ and $Aax\neq 0$, but since $r(b)=r(a)$, this is contradiction.

2) C-3) $\to$ C-1).

i) $r(N)=l(N)$. This follows easily from our Lemma.

ii) Let Aa and Ab are two simple left ideals and $r(a)=r(b)$, then $Aa=Ab$.

Now $aA\cong bA$ (they are simple right ideals by Lemma) and we can assume that a corresponds to b, whence there exists x such that $b=xa$. Thus $Ab=Axa=Aa$.

iii) $Ae_{\kappa,i}$ has a unique simple sub-ideal, and the same holds for $e_{\kappa,i}A$ for every κ, i.

For, simple left sub-ideals in $Ae_{\kappa,i}$ have the same maximal right ideal $\sum_{\tau\neq\kappa,\, S\neq i} e_{\tau,S}A\smile N$ as their right annihilators, by i). Therefore they are equal, by ii). Thus we have proved by i) of 1).

About C-1) $\rightarrow$ C-2) and C-2) $\rightarrow$ C-4′), we have to refer [1] and [2].

Remark. We can also deduce directly C-3) $\rightarrow$ C-2).

In fact: If $\mathfrak{l}$ and $\mathfrak{l}'$ are two left ideals such that $\mathfrak{l}\supsetneqq\mathfrak{l}'$, $\mathfrak{l}'$ is maximal in $\mathfrak{l}$ and $r(\mathfrak{l}')=Q\supsetneqq r(\mathfrak{l})=Q'$, then we can write $\mathfrak{l}=Aa\smile\mathfrak{l}'$, whence $Q'=r(a)\frown Q$, and $aQ\cong Q/r(a)\frown Q=Q/Q'$.

Let x be an element such that $x\in Q$, $\notin Q'$, then $\mathfrak{l}x=(Aa\smile\mathfrak{l}')x =Aax=Aax/\mathfrak{l}'x$.

Now let $\mathfrak{l}/\mathfrak{l}'=Aa\smile\mathfrak{l}'/\mathfrak{l}'\cong Aa/Aa\frown\mathfrak{l}'\cong\bar{A}\bar{e}_\kappa\cong Ab$, and a corresponds to b for suitable e_κ and b, and let $Q''=l(b)$ (maximal left ideal), then $Q''a\subseteqq\mathfrak{l}'$, therefore $Q''ax\subseteqq\mathfrak{l}'x=0$.

That is, Aax is a simple left ideal. Furthermore since ax and ax' have the same left annihilator Q'' for any two elements x and x', $\in Q$, $\notin Q'$, $Aax\cong Aax'$. Therefore $axA=ax'A=aQ$ and they are simple right ideals.

Thus Q' is maximal in Q.

Thus we have proved that the composition lengths of right and left ideals are equal, and therefore annihilator relations hold.

References

1) T. Nakayama: On Frobeniusean Algebra II. Ann. Math., **42** (1942).

2) M. Ikeda: A Characterization of Quasi-Frobenius Rings. Osaka Math. J., **4**, No. 2 (1952).