58. A Proof of the Plancherel Theorem

By Hisaaki YOSHIZAWA
Department of Mathematics, Osaka University
(Comm. by K. KUNUGI, M.J.A., April 12, 1954)

1. **Introduction.** The purpose of this note is to give a simple proof to the generalized Plancherel theorem. This is one of the fundamental theorems in harmonic analysis on commutative topological groups, and various formulations and proofs to it have been published. We follow essentially the formulation given by Godement in his paper [2] (see the references at the end of the note); that is, we formulate the theorem as a proposition on realization of a certain kind of representation of a self-adjoint algebra. We shall give the exact statement in section 3, and the proof in section 4. The key idea of our proof is consideration of the so-called approximate identity. In section 2 we shall give definitions of some general notions which are necessary to state the theorem. In the last section, which concerns the special case of commutative groups, some remarks will be given on the Khintchine theorem on approximation of a positive definite function by the convolution of a square integrable function with its adjoint.

2. **(H)-representation.** Let $\mathbf{A}$ be a *$*$-algebra*, that is, a complex algebra which admits an adjoint operation $x \to x^*$. By a *representation* of $\mathbf{A}$ we mean a representation of the $*$-algebra $\mathbf{A}$ by bounded linear operators on a Hilbert space $\mathbf{H}$: $x \to R(x)$, where the adjoint of an operator is to be defined as the usual one. If, moreover, there corresponds to each element x in $\mathbf{A}$ an element in $\mathbf{H}$, which we designate by $\Phi(x)$ or $\dot{x}$, in such a manner that the following three conditions are satisfied, then we call the triplet $\{\mathbf{H}, R, \Phi\}$ an *(H)-representation* of $\mathbf{A}$:

(1) Φ is linear.

(2) $(xy)^{\cdot} = R(x)\dot{y}$ for $x \in \mathbf{A}$, $y \in \mathbf{A}$.

(3) The image of the whole $\mathbf{A}$ by Φ is everywhere dense in $\mathbf{H}$.

An (H)-representation is called *proper* if for any nonzero element φ in $\mathbf{H}$ there exists an x in $\mathbf{A}$ such that $R(x)\varphi \neq 0$. A filtre $\{u\}$ whose elements u belong to $\mathbf{A}$ is called an *approximate identity* with respect to the given (H)-representation, if, for each $x \in \mathbf{A}$, $\{u\dot{x}\}$ converges to $\dot{x}$ strongly in $\mathbf{H}$ according to the filtre $\{u\}$. (Every (H)-representation of $\mathbf{A}$ corresponds to a bilinear functional on $\mathbf{A} \times \mathbf{A}$ which satisfies certain simple conditions, and this correspondence may be considered as a general formulation of the method of utilizing positive functionals in the theory of so-called unitary representation. But in this note we discuss these matters no further.)

3. **The Plancherel theorem.** First we shall give preliminaries for the Plancherel theorem. They are familiar in literatures (see, in particular, Godement [2]), but we shall mention them for clarity. Let $\mathbf{A}$ be a commutative $*$-algebra of bounded operators on a Hilbert space. A character χ of $\mathbf{A}$ is a linear, Hermitian ($\chi(x^*)=\overline{\chi}(x)$) and multiplicative ($\chi(xy)=\chi(x)\chi(y)$) functional on $\mathbf{A}$ which is not identically zero and continuous in the uniform topology. For any algebra $\mathbf{M}$ of operators we shall designate by $\overline{\mathbf{M}}$ the uniform closure of $\mathbf{M}$ and by $\mathbf{M}^1$ the algebra which is algebraically generated by $\mathbf{M}$ and the identity operator. Then the *space of characters* $\Omega(\mathbf{A})$ of $\mathbf{A}$, which is, by definition, the set of all characters of $\mathbf{A}$ topologized by the weak topology, may be identified with the space $\Omega(\overline{\mathbf{A}^1})$, with possible exception of the *infinity point* of the latter. The Fourier transform $T^\sim$ of an element T in $\overline{\mathbf{A}}^1$: $T^\sim(\chi)=\chi(T)$ is continuous on $\Omega(\mathbf{A})$; if T is in $\mathbf{A}$, its transform vanishes at infinity. Conversely every continuous function on $\Omega(\mathbf{A})$ which vanishes at infinity is identical with the Fourier transform of some element of $\overline{\mathbf{A}}$. By the Fourier transformation, $\mathbf{A}$ is isomorphic, as a normed $*$-algebra, with the algebra consisting of its Fourier transforms, the norm of a continuous function ξ being defined by $\|\xi\|=\sup\xi(\chi)$. Now let $\mathbf{A}$ be an abstract commutative $*$-algebra and R be a representation of $\mathbf{A}$. We call $\Omega(R(\mathbf{A}))$ the space of characters of $\mathbf{A}$ with respect to the representation R (where $R(\mathbf{A})$ stands for the algebra consisting of operators $R(x)$, $x\in\mathbf{A}$), and the function $(R(x))^\sim$ on $\Omega(R(\mathbf{A}))$ the Fourier transform of x with respect to R. Using these notions and notations we formulate the Plancherel theorem as follows:

THEOREM. *Let* $\mathbf{A}$ *be a commutative* $*$*-algebra and* $\{\mathbf{H}, R, \Phi\}$ *be a proper* (H)*-representation of* $\mathbf{A}$. *Denote the space of characters of* $\mathbf{A}$ *with respect to the representation* R *by* Ω *and the corresponding Fourier transform of* x *in* $\mathbf{A}$ *by* $\tilde{x}$. *Then:*

(a) *There exists a measure* μ *on* Ω *which is uniquely determined, nonzero for open sets and finite for compact sets.*

(b) *The Fourier transforms of all elements in* $\mathbf{A}$ *belong to* $L^2(\Omega,\mu)$ *and they are everywhere dense in it.*

(c) *The inner product in the Hilbert space* $\mathbf{H}$ *is expressed by the integral over* Ω *with respect to* μ; *that is, for any* x *and* y *in* $\mathbf{A}$ *and for any operator* T *in the algebra* $\overline{R(\mathbf{A})}^1$, *we have*

$$\langle T\dot{x}, \dot{y}\rangle = \int \tilde{T}(\chi)\tilde{x}(\chi)\overline{\tilde{y}(\chi)}d\chi.$$

(c′) *In particular, for any x and y in* **A** *we have*

$$\langle \dot{x}, y \rangle = \int \tilde{x}(\chi)\overline{\tilde{y}(\chi)}d\chi.$$

4. Proof. We shall divide our proof into four steps.

FIRST STEP. It is sufficient to prove the assertions for the algebra $R(\mathbf{A})$ rather than for **A**; that is, we shall assume that the algebra **A** in the theorem is itself a (normed) algebra of operators and the representation R is the identical one: $R(x)=x$. Indeed, it is easy to show that the mapping $R(x)\to x$ satisfies the conditions required in the definition of proper (H)-representation by the assumption of properness of the given (H)-representation of **A**. Hence in the following we shall not distinguish $R(\mathbf{A})$ from **A**.

SECOND STEP. We shall construct a suitable algebra **B** of operators such that **B** contains **A** and every continuous function on Ω with compact carrier is identical with the Fourier transform of some element in **B**. For this purpose, we define **B** as the ideal of $\overline{\mathbf{A}}^1$ which is finitely generated by **A**:

$$\mathbf{B}=\left\{T_1x_1+\cdots+T_nx_n;\quad T_k\in\overline{\mathbf{A}}^1,\ x_k\in\mathbf{A}\right\}.$$

Then, the mapping Φ (of the given (H)-representation) from **A** into **H** is extended to a mapping defined on **B** (we denote it by the same notation Φ) by putting

$$\Phi(\sum T_kx_k)=\sum T_k\Phi(x_k).$$

Indeed, this definition is possible: If $\sum T\dot{x}\neq 0$, there exists a y in **A** such that $y(\sum Tx)\dot{y}\neq 0$ in **H**, which means that $\sum Tx\neq 0$ in **B**. Now the Fourier transform **B**~ of **B** contains every continuous function with compact carrier since the transform of **A** contains all continuous functions on Ω.

In order to prove the theorem, it is sufficient to prove assertions (a), (b) and (c′) for the algebra **B** rather than for **A**. Indeed, firstly the space of characters Ω is unaltered in this replacement, since obviously $\overline{\mathbf{A}}=\overline{\mathbf{B}}$. Secondly **A**·, the set of all $\dot{x}$ for $x\in\mathbf{A}$, is everywhere dense in **H** by the assumption, and so in **B**· too; hence if we show that **B**~ is dense in $L^2(\Omega)$, it will follow, under assertion (c′) proved for **B**, that **A**~ is also dense in $L^2(\Omega)$. Now continuous functions with compact carriers are everywhere dense in $L^2(\Omega)$ by themselves; hence assertion (b) will be proved for **B** if we show that **B**~ is contained in $L^2(\Omega)$, or, a fortiori, assertion (c′) for **B**. Therefore it is sufficient to prove assertions (a) and (c′) under the assumption that the algebra **A** is such that its transform **A**~ contains every continuous function with compact carrier.

THIRD STEP. We shall show that **A** contains an approximate identity. For this purpose we introduce ordering in the set of all

Hermitian elements ($x^* = x$) in **A**, as usual, by putting $x \geqq y$ when $x - y$ is a positive or zero operator, that is, $\tilde{x}(\chi) - \tilde{y}(\Omega) \geqq 0$ on Ω.

We call a subset $U = \{u\}$ of **A** a *U-filter* if it satisfies the following three conditions:

(4) $0 \leqq \tilde{u}(\chi) \leqq 1$ for $u \in U$.

(5) U constitutes a filtre in the ordering defined above.

(6) $\tilde{u}(\chi)$ converges to 1 at every $\chi \in \Omega$.

It follows that $\tilde{u}(\chi)$ converges to 1 uniformly on compact sets. Now we denote by U_0 the set consisting of all those u whose Fourier transforms have compact carriers and satisfy condition (4). Then U_0 is a U-filtre; it will be utilized later to conclude the proof. (More generally we can show that **A** contains a U-filtre in any case, that is, without the assumption that **A**~ contains every function with compact carrier. But we do not use this fact in our proof.) Now we shall show that any U-filtre $U = \{u\}$ is an approximate identity. For any fixed x, $\{u\dot{x}\}$ is a Cauchy filtre in **H** (according to the filtre U): For, first it is a bounded set in **H** and secondly for any pair u, v in U which satisfies $u \leqq v$, we have

$$| v\dot{x} - u\dot{x} |^2 \leqq | v\dot{x} |^2 - | u\dot{x} |^2$$

by a usual argument. Let ξ be the limit point of $\{u\dot{x}\}$. Then we have, for every $y \in \mathbf{A}$, both $yu\dot{x} \to y\xi$ and $yu\dot{x} \to y\dot{x}$; the latter convergence is valid since $\tilde{y}(\chi)\tilde{u}(\chi)$ converges to $\tilde{y}(\chi)$ uniformly on Ω, or $yu \to y$ in **A**. Hence by the assumption of properness we have necessarily $\xi = \dot{x}$.

FOURTH STEP. Put

$$\Gamma = \left\{ (y^*x)^\sim ;\ x \in \mathbf{A},\ y \in \mathbf{A} \right\};$$

then Γ contains every continuous function on Ω with compact carrier. Now let $U = \{u\}$ be an approximate identity; then we have for $x \in \mathbf{A}$, $y \in \mathbf{A}$,

$$(7) \qquad \langle \dot{x}, \dot{y} \rangle = \lim \langle x\dot{u}, \dot{y} \rangle : \ (u \in U),$$

$$(8) \qquad \langle \dot{x}, \dot{y} \rangle = \lim \langle y^*xu, \dot{u} \rangle : \ (u \in U).$$

Expression (8) permits us to define a functional μ on Γ as follows:

$$\mu((y^*x)^\sim) = \langle \dot{x}, \dot{y} \rangle.$$

It is easy to verify that μ satisfies the following three conditions for $\xi \in \Gamma$, $\eta \in \Gamma$:

(9) If $\xi + \eta \in \Gamma$, then $\mu(\xi + \eta) = \mu(\xi) + \mu(\eta)$.

(10) $\mu(\alpha\xi) = \alpha\mu(\xi)$.

(11) If $\xi(\chi) \geqq 0$ on Ω, then $\mu(\xi) \geqq 0$.

Taking, as U in identity (7), the special approximate identity U_0 defined above (the Fourier transforms of whose elements have compact carriers), we have

(12) $\mu(\xi) = \lim \mu(\xi \cdot \tilde{u})$: $(u \in U_0)$, where $\xi \cdot \tilde{u}$ means the ordinary product

of functions $\tilde{\xi}(\chi)$ and $\tilde{u}(\chi)$.
By properties (9)–(12), the inner product $\langle \dot{x}, \dot{y} \rangle$ is expressed by the integral over Ω with respect to a measure μ:

$$\langle \dot{x}, \dot{y} \rangle = \int (y^* x)^{\sim}(\chi) d\chi,$$

which proves assertion (c′) of the theorem. The properties of the measure μ mentioned in (a) are easily verified. This completes the proof of the theorem in its general form.

5. Approximation of positive definite functions. In order to apply the Plancherel theorem in section 3 to a commutative group G, it is sufficient to consider the group algebra $\mathbf{A}$ of G (multiplication being defined by convolution) consisting of all continuous functions on G with compact carriers. Then $L^2(G)$, convolution and embedding of $\mathbf{A}$ into $L^2(G)$ will play the roles of $\mathbf{H}$, R and Φ respectively, and obviously the group algebra $\mathbf{A}$ may be regarded an algebra of operators on $L^2(G)$: $\mathbf{A} = R(\mathbf{A})$. In this specialization it is desirable to show that the character group $G^\wedge$ coincides with the space of characters Ω of $\mathbf{A}$ (with respect to the regular representation of $\mathbf{A}$) by the natural correspondence between these two sorts of characters: $G^\wedge = \Omega$. Since it is obvious that the relation of inclusion holds: $G^\wedge \supseteqq \Omega$, together with equivalency of topologies in these two spaces, we shall show that every character (in the ordinary sense) of G defines a continuous character on the algebra $\mathbf{A}$ of operators. For this purpose it is sufficient to show that if a character χ_0 on G defines a (continuous) character on $\mathbf{A}$ (the existence of such a character is guaranteed in the abstract theorem in section 3), then the (ordinary) product $\chi_0(s)\chi(s)$ with any character χ of G defines also a character on $\mathbf{A}$ in the same sense. This fact follows easily from the consideration in the following. But we can show it also by simple computation of norms of functionals and operators. (This direct way was noticed to me by H. Umegaki.)

In relation with the preceding consideration, we shall add a few remarks, modifying some problems which were discussed in Godement [1]. Let G be a (not necessarily commutative) group possessing a two-sided invariant measure, and let f be a continuous positive definite function on G. Then the following three conditions are mutually equivalent:

(13) The functional defined by f on the group algebra $\mathbf{A}$ is continuous in the operator norm of $\mathbf{A}$.

(14) For any integrable positive definite function φ, we have

$$\int_G f(s)\varphi(s) ds \geqq 0.$$

(15) There exists a filtre $\{x\}$ whose elements x are square integrable

functions on G, and we have

$$x^*x(s) \to f(s)$$

uniformly on compact sets, where the multiplication means convolution.

In general, these properties are possessed not necessarily by all positive definite functions on a given group, as was shown in my previous paper [3]. But for a commutative group, thus we obtain a proof to the Khintchine theorem which asserts that every positive definite function is approximated as in condition (15),— a proof which is not based on the Plancherel theorem for groups.

We shall discuss the problems in this section further and more closely on another occasion.

References

[1] Godement, R.: Les fonctions de type positif et la théorie des groupes, Trans. Amer. Math. Soc., **63**, 1–84 (1948).

[2] ——: Sur la théorie des representations unitaires, Ann. Math., **53**, 68–124 (1951).

[3] Yoshizawa, H.: Some remarks on unitary representations of the free group, Osaka Math. J., **3**, 55–63 (1951).