# 74. *On Multiple Distributions*

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Equations of evolution have been discussed by several authors, but it seems to me that researches have been done generally with respect to parametric operatorial equations or parametric distributional equations and scarcely with respect to proper distributional equations. So as a first step to the researches of the latter it will be of some interest to consider the general relations between parametric and proper distributional equations. To give a clarification of this relation we introduced the notion of multiple distributions defined in §3. At the same time the study of our multiple distributions will be helpful for the construction of resolvants of proper distributional equations.

Some other problems have also close relation to the study of multiple distributions, say, multiplication by particular distribution, the distributional treatment of $\Delta$-type functions or of $S$-matrix. In this paper discussions of these problems are not stated, however, §5, §6, and §3 have relations to some parts of them.

1. First we modify a few B. H. Arnold's results. Let $S=\{\theta, x, y\dots\}$ be a vector space over the real number field with zero vector $\theta$, and $\mathfrak{B}$ be any collection of subsets of $S$ satisfying the following axioms;

(B1) For any $x\in S$, $\{x\}\in\mathfrak{B}$.
(B2) The union of any two sets of $\mathfrak{B}$ is a set of $\mathfrak{B}$.
(B3) Any subset of a set of $\mathfrak{B}$ is a set of $\mathfrak{B}$.
(B4) Any scaler multiple of a set of $\mathfrak{B}$ is a set of $\mathfrak{B}$.
(B5) The convex hull of a set of $\mathfrak{B}$ is a set of $\mathfrak{B}$.

We call the elements of $\mathfrak{B}$ bounded subsets of the vector space $S$.

*Definition 1. A subset $G$ of $S$ is called open if and only if whenever $g\in G$, there exists a convex set $N$ such that for any $B\in\mathfrak{B}$ there exists a $\lambda>0$ which satisfies $g+\lambda B\subset N\subset G$.*

A set $T\subset S$ is called topologically bounded if and only if for each neighborhood $U$ of $\theta$ there exists a $\lambda$ with $T\subset\lambda U$. We denote by $\mathfrak{T}$ the collection of all subsets of $S$ which are topologically bounded.

**Lemma 1.** *$\mathfrak{T}\supset\mathfrak{B}$, and the collection $\mathfrak{T}$ satisfies axioms from B1) to B5).*

**Lemma 2.** *The topologies defined in $S$ by the collection $\mathfrak{T}(\tau_{\mathfrak{T}})$ and by the collection $\mathfrak{B}(\tau_{\mathfrak{B}})$ are identical.*

**Theorem 1.** *Definition 1 makes $S$ a locally convex bornographic topological vector space.*

2. We consider the vector space $\mathfrak{D}$ of all real valued infinitely differentiable functions having compact carriers. We denote the space $\mathfrak{D}$ defined on the $n$-dimensional Euclidean space $R^n(x)$ by $\mathfrak{D}(x)$, similarly the one on $R^m(t)$ by $\mathfrak{D}(t)$ and the one on $R^{m+n}(x,t)$ by $\mathfrak{D}(x,t)$, where $m>0$ and $n\geq 0$. We use also notations $\mathfrak{D}^{(\nu)}(x)$, $\mathfrak{D}^{(\nu)}(t)$, $\mathfrak{D}^{(\nu)}(x,t)$ for the space of all $\nu$-times continuously differentiable functions having compact carriers defined on $R^n(x)$, $R^m(t)$, $R^{m+n}(x,t)$ and denote the totality of bounded sets in their natural topology by $\mathfrak{B}_{\mathfrak{D}}(t)$, $\mathfrak{B}_{\mathfrak{D}^\nu}(t)$, etc. Further we denote their strong dual spaces by $\mathfrak{D}^{(\nu)\prime}(x)$, $\mathfrak{D}^{(\nu)\prime}(t)$ etc., and denote the convergence in the topology of $\mathfrak{D}^{(\nu)\prime}$ by the symbol $\xrightarrow{(\nu)'}$.

Now we take a sequence of functions $\{\phi_j(t)| \in \mathfrak{D}(t), \xrightarrow{(\nu)'} Q\}$ where $Q$ is a definite distribution of $\mathfrak{D}^{(\mu)\prime}(t)$ for $\mu\leq\nu$, and often call it a $(\nu, Q)$*-sequence*. We take the totality of such sequences and denote it by $\mathfrak{B}'(t)$, i.e. $\mathfrak{B}'(t)=\{B'_\alpha | B'_\alpha=\{\phi_{\alpha j}\}, \phi_{\alpha j}\xrightarrow{(\nu)'} Q\}$ and consider the minimum collection of subsets of $\mathfrak{D}(t)$ which satisfies axioms from B1) to B5) and includes both $\mathfrak{B}'(t)$ and $\mathfrak{B}_{\mathfrak{D}}(t)$. For the sake of simplicity we call such a collection an *Arnold's family*. In this case such an Arnold family $\mathfrak{B}^0(t)$ really exists and uniquely determined and is given by a collection of sets of the following form;

$$\mathfrak{B}^0(t)=\{B^0(t)=((B(t)\cup U_{i=1}^k \lambda_i B_i'(t)))|B\in\mathfrak{B}_{\mathfrak{D}}(t), B'\in\mathfrak{B}'(t), k=1,2,\dots\}$$

where the symbol $((A))$ means the convex hull of a set $A$. We denote by $\mathfrak{N}(t)$ a fundamental neighborhood system of $\theta$ which is induced by $\mathfrak{B}^0(t)$ obeying the method §1, its element by $N(t)$, and denote the space $\mathfrak{D}(t)$ having this topology by $\mathfrak{D}_Q(t)$.

Next we consider the tensor product space $\mathfrak{D}(x)\otimes\mathfrak{D}(t)$, i.e.

$$\mathfrak{D}(x)\otimes\mathfrak{D}(t)=\{\textstyle\sum_i\varphi_i(x)\phi_i(t)|\varphi_i\in\mathfrak{D}(x),\ \phi_i\in\mathfrak{D}(t)\},$$

where $\sum_i$ means finite linear sum. We consider here the Arnold's family $\mathfrak{B}^0(x,t)$ which includes a family of subsets

$$\{B(x)\otimes B^0(t)|B\in\mathfrak{B}_{\mathfrak{D}}(t),\ B^0\in\mathfrak{B}^0(t)\},$$

where

$$B(x)\otimes B^0(t)=\{\varphi(x)\phi(t)|\varphi\in B(x),\ \phi\in B^0(t)\}.$$

Then $\mathfrak{B}^0(x,t)$ is also uniquely determined and is given by the collection of subsets $\{((B(x)\otimes B^0(t)))|B\in\mathfrak{B}(x),\ B^0\in\mathfrak{B}^0(t)$, and their subsets$\}$. We denote a fundamental neighborhood system of $\theta$ which this family induces by $\mathfrak{N}(x,t)=\{N(x,t)\}$.

We find similarly in the space $\mathfrak{D}(x,t)$, the Arnold's family which includes $\mathfrak{B}^0(x,t)$ and $\mathfrak{B}_{\mathfrak{D}}(x,t)$ i.e.,

$\{((B(x,t) \cup B^0(x,t)))| B(x,t) \in \mathfrak{B}_{\mathfrak{D}}(x,t),\ B^0(x,t) \in \mathfrak{B}^0(x,t)\}$.
A fundamental neighborhood system of $\theta$ is given by

$$\{((V(x,t) \cup N(x,t)))| V \in \mathfrak{V}(x,t),\ N \in \mathfrak{N}(x,t)\},$$

where $\mathfrak{V}(x,t)$ means a fundamental neighborhood system of $\theta$ in the natural topology of $\mathfrak{D}(x,t)$. We denote the space $\mathfrak{D}(x,t)$ having this topology by $\mathfrak{D}_Q(x,t)$ or simply by $\mathfrak{D}_Q$.

Thus $\mathfrak{D}_Q(x,t)$ is introduced by a single distribution $Q$, but a similar process is possible for a fixed family of distributions $\{Q_\lambda | \lambda \in \Lambda\}$. That is to say $B^0(t)$ is expressed by $B^0(t) = ((B(t) \cup \bigcup_{k=1}^{s} \bigcup_{i=1}^{r} \bigcup_{j=1}^{\infty} \mu_{ik} \phi_{ij\lambda_k}(t)))$, where $\phi_{ij\lambda_k} \xrightarrow{(\nu)'} Q_{\lambda_k}$, and of course $\nu$ is larger than the orders $\mu$ of distribution $Q_{\lambda_k}$. The forms of Arnold's family in the other spaces, say, $\mathfrak{D}(x) \otimes \mathfrak{D}(t)$ and $\mathfrak{D}(x,t)$ is quite similar. We denote the space $\mathfrak{D}(t)$ or $\mathfrak{D}(x,t)$ having this topology by $\mathfrak{D}_Q(t)$ or $\mathfrak{D}_{Q_\Lambda}(x,t)$. The orders $\mu$ of the distribution $Q_\lambda$ and the orders of the convergence $\nu = \nu(\lambda)$ can be various, but we have interest only in the case when both $\mu$ and $\nu$ are constants, and consider only this case. From the same family of distributions $\{Q_\lambda\}$, we can also construct another $\mathfrak{B}^0(t)$ i.e., we modify a bounded set $B^0(t)$ as following.

Taking a family of sequences such that

$$\left\{ \begin{array}{l|l} \{\phi_{\lambda j} | j\} & \text{For any neighborhood of } \theta \text{ of } \mathfrak{D}^{(\nu)'}(t),\ V, \text{ there exists} \\ & j_0 \text{ such that (i) for any } j \geq j_0, \text{ for any } \lambda \in \Lambda,\ \phi_{\lambda j} - Q_\lambda \in V \\ \lambda \in \Lambda & \text{(ii) } \bigcup_{j<j_0, \lambda \in \Lambda} \phi_{\lambda j} \in \mathfrak{B}_{\mathfrak{D}}(t), \end{array} \right\}$$

we call it a $(\nu, \tilde{Q})$ *family* and write this element by $B'_i(t)$. Now we consider $B^0(t) = ((B(t) \cup \bigcup_{i=1}^{k} \rho_i B'_i(t)))$ or its arbitrary subset. The other forms are quite the same. We denote the space $\mathfrak{D}(t)$ or $\mathfrak{D}(x,t)$ having this topology by $\tilde{\mathfrak{D}}_Q(t)$ or $\tilde{\mathfrak{D}}_Q(x,t)$. We often consider properties common to each of the spaces $\mathfrak{D}_Q(x,t)$, $\mathfrak{D}_{Q_\Lambda}(x,t)$, $\tilde{\mathfrak{D}}_Q(x,t)$. In such a case we denote them collectively by $\mathfrak{D}_P$, similarly denote $\mathfrak{D}_Q(t)$, $\mathfrak{D}_{Q_\Lambda}(t)$, $\tilde{\mathfrak{D}}_Q(t)$ by $\mathfrak{D}_P(t)$.

**Lemma 3.** *For any neighborhood $N(x,t)$ of $\theta$ in $\mathfrak{D}(x) \otimes \mathfrak{D}(t)$ and for any bounded set $B(x)$, there exists a neighborhood of $\theta$ in $\mathfrak{D}(x)$ such that $N(x,t) \supset V(x) \otimes B^0(t)$. Similarly for any bounded set $B^0(t)$ there exists a neighborhood of $\theta$, $V(x)$, in $\mathfrak{D}(x)$ such that $N(x,t) \supset V(x) \otimes B^0(t)$.*

**Corollary.** *$T \in \mathfrak{D}'_P(x,t)$ is separately continuous for $\mathfrak{D}(x)$ and $\mathfrak{D}_P(t)$.*

3. We consider the strong dual space $\mathfrak{D}'_P$ of $\mathfrak{D}_P$ and the closure of the space $\mathfrak{D}$ in the topology of $\mathfrak{D}'_P$, and denote this closure by $'\mathfrak{D}_P$. If $T \in {'\mathfrak{D}_Q}$ we have a filter $\mathfrak{F}$ on $\mathfrak{D}(x,t)$ such that $\mathfrak{F} \xrightarrow{\mathfrak{D}'_Q} T$ and we can prove $\langle f, \varphi Q \rangle$ converges with respect to $\mathfrak{F} \xrightarrow{\mathfrak{D}'_Q} T$ uniformly

for $\varphi \in \mathfrak{B}_{\mathfrak{D}}(x)$ and independently of $\mathfrak{F}$ which converges to $T$ in the topology of $\mathfrak{D}'_Q$.

*Definition 2. Multiple distribution of a distribution $T \in \mathfrak{D}'(x, t)$ by a distribution $Q \in \mathfrak{D}'(t)$ is a distribution $T_Q \in \mathfrak{D}'(x)$ such that for $\varphi \in \mathfrak{D}(x)$, $\langle T_Q, \varphi\rangle = lim^{\mathfrak{D}'_Q}_{\mathfrak{F}\to T}\langle f, \varphi Q\rangle$.*

**Theorem 2.** *If $T \in {}'\mathfrak{D}_Q$, then $T$ is continuous uniformly for $\varphi \in B(x)$ with respect to any $(\nu, \rho Q)$ sequence, where $\rho$ is an arbitrary constant and this limit coincides with $\rho T_Q$.*

**Corollary.** *If $T \in {}'\tilde{\mathfrak{D}}_Q$ and $[\{\phi_{\lambda j}\} | \lambda \in \Lambda]$ is a $(\nu, \rho\tilde{Q})$ family, then $T$ is continuous with respect to the sequence $\{\phi_{\lambda j} | j = 1, 2, \ldots\}$ uniformly for $\lambda \in \Lambda$ and uniformly for $\varphi \in B(x)$.*

**Theorem 3.** *If $T \in \mathfrak{D}'(x, t)$ and $\langle T, \varphi\phi_j\rangle$ makes a Cauchy sequence with respect to any $(\nu, \rho Q)$ sequence $\{\phi_j\}$ for $\rho = 0, 1$, uniformly for $\varphi \in B(x)$, then $T \in {}'\mathfrak{D}_Q$.*

**Corollary.** *If $T \in \mathfrak{D}'(x, t)$ and $\{\langle T, \varphi\phi_{\lambda j}\rangle | j = 1, 2, \ldots\}$ makes a Cauchy sequence with respect to any $\{\phi_{\lambda j}\}$ of a $(\nu, \rho\tilde{Q})$ family $[\{\phi_{\lambda j}\} | \lambda \in \Lambda]$ for $\rho = 0, 1$, uniformly for $\varphi \in B(x)$ and uniformly for $\lambda \in \Lambda$, then $T \in {}'\tilde{\mathfrak{D}}_Q$.*

4. **Theorem 4.** *The mapping $T \longrightarrow T_Q$ is a continuous linear mapping from ${}'\mathfrak{D}_P(x, t)$ to $\mathfrak{D}'(x)$.*

We denote a differential operator in $R^m(t)$ such as $\sum_{|s| \le \mu} a_{s_1 \ldots s_m} \partial^{|s|}/\partial t_1^{s_1} \ldots \partial t_m^{s_m}$, where $|s| = s_1 + \cdots + s_m$ and $a_{s_1 \ldots s_m}$ is a constant by $D_t^\mu$ and its conjugate operater by $D_t^{\mu *}$ i.e. $D_t^{\mu *} = \sum_{|s| \le \mu} (-1)^{|s|} a_{s_1 \ldots s_m} \partial^{|s|}/\partial t_1^{s_1} \ldots \partial t_m^{s_m}$.

**Theorem 5.** *If $T \in {}'\mathfrak{D}_{D_t P}$, then $D_t^* T \in {}'\mathfrak{D}_P$ and $T_{D_t Q_\lambda} = (D_t^* T)_{Q_\lambda}$. Especially if $D_t^\mu$ is a product such that $D_t^\mu = D_t^{\mu_1} D_t^{\mu_2}$, then from $T \in \mathfrak{D}'_{D_t^\mu P}$ it follows that $D_t^{\mu_1 *} T \in {}'\mathfrak{D}_{D_t^{\mu_2} P}$ and $T_{D_t^\mu Q} = (D_t^{\mu_1 *} T)_{D_t^{\mu_2} Q}$.*

The theorem may be stated more generally. Now we consider a mapping $L_t$ of $\mathfrak{D}(t)$ into itself which satisfies the following conditions. (i) $L_t$ maps any $(\nu, \rho Q_\lambda)$ sequence to a $(\nu', \rho' L_t Q_\lambda)$ sequence or maps any $(\nu, \rho\tilde{Q})$ family to a $(\nu', \rho'\widetilde{L_t Q})$ family for $\rho = 0, 1$, and arbitrary constant $\rho'$. (ii) $L^*(\mathfrak{D}) \subset \mathfrak{D}$, where $L^*$ is a conjugate operator of $\mathfrak{D}'(x, t)$ into itself defined by $\langle L^* T, \varphi\phi\rangle = \langle T, \varphi(L_t\phi)\rangle$ for $\varphi \in \mathfrak{D}(x)$, $\phi \in \mathfrak{D}(t)$. Concerning this mapping $L_t$ the following lemma holds.

**Lemma 4.** *If $T \in {}'\mathfrak{D}_{L_t P}$, then $L^* T \in {}'\mathfrak{D}_P$ and $T_{L_t Q_\lambda} = (L^* T)_{Q_\lambda}$.*

**Theorem 6.** *If the topology of ${}'\mathfrak{D}_{D_t^\mu P}$ is introduced by bounded sets such that every $(\nu + \mu, \rho D_t^\mu Q_\lambda)$ sequence or $(\nu + \mu, \rho\widetilde{\mathfrak{D}_t^\mu Q})$ family for $\rho = 0, 1$ is a map by $D_t^\mu$ of a $(\nu, \rho' Q)$ sequence or $(\nu + \mu, \rho'\tilde{Q})$ family respectively and if $D^* T \in {}'\mathfrak{D}_P$ then we have $T \in {}'\mathfrak{D}_{D_t^\mu P}$.*

**Theorem 7.** *If $T \in {}'\tilde{\mathfrak{D}}_Q$ and $\Lambda = \{\lambda\}$ is a topological space and* *the mapping* $\lambda \longrightarrow Q_\lambda$ *is continuous as the mapping from* $\Lambda$ *into* $\mathfrak{D}^{(\mu)\prime}(t)$, *then the mapping* $\lambda \longrightarrow T_{Q_\lambda}$ *is a continuous mapping from* $\Lambda$ *to* $\mathfrak{D}'(x)$.

**Lemma 5.** *If* $T \in {}'\mathfrak{D}_P$, $S \in \mathfrak{E}'(x)$, *then* $(S \times \delta(t)) * T \in {}'\mathfrak{D}_P$ *and* $\{(S \times \delta(t)) * T\}_{Q_\lambda} = S \underset{(x)}{*} T_{Q_\lambda}$.

**Corollary.** *If* $T \in {}'\mathfrak{D}_P(x, t)$ *then* $D_x^p T \in {}'\mathfrak{D}_P$ *and* $(D_x^p T)_{Q_\lambda} = D_x^p T_{Q_\lambda}$.

**Lemma 6.** *If* $T \in {}'\mathfrak{D}_P(x, t)$, $f(t) \in \mathfrak{E}(t)$, $g(x) \in \mathfrak{E}(x)$, *then* $f(t)g(x)T \in {}'\mathfrak{D}_P$ *and* $(f(t) \cdot g(x) \cdot T)_{Q_\lambda} = g(x) \cdot T_{f.Q_\lambda}$.

5. Hereafter we confine ourselves to some special cases. We take Dirac's $\delta$ and its $\mu$-th derivative $\delta^{(\mu)}$ as $Q$, and $t$ itself as $\lambda$ and $D_t$ as $L_t$. We treat only the case $m$ is 1 though quite similar results are obtained in the case $m \neq 1$ too. We take an interval $V$; $a \leq t \leq b$, as $\Lambda$. Further we write ${}'\mathfrak{D}_{t_0(\mu)}$ in the place of ${}'\mathfrak{D}_{\delta_{t_0}(\mu)}$ and ${}'\mathfrak{D}_{\mathfrak{V}(\mu)}$ in the place of ${}'\mathfrak{D}_{\delta_\Lambda(\mu)}$, similarly ${}'\widetilde{\mathfrak{D}}_{\mathfrak{V}(\mu)}$, and ${}'\mathfrak{D}_{t_0}$ for ${}'\mathfrak{D}_{\delta_{t_0}(0)}$ and ${}'\mathfrak{D}_{\mathfrak{V}}$ for ${}'\mathfrak{D}_{\mathfrak{V}(0)}$, ${}'\widetilde{\mathfrak{D}}_{\mathfrak{V}}$ for ${}'\widetilde{\mathfrak{D}}_{\mathfrak{V}(0)}$. We use also notations $\partial^\mu T/\partial t_0$ in the place of $T_{\delta_t(\mu)}$ and $T_t$ for $T_{\delta_t}$. These designations are not so unreasonable, since for example if $T = f(t)S(x)$ where $f(t) \in \mathfrak{D}^{(\mu)}(t)$ and $S(x) \in \mathfrak{D}'(x)$ then $T \in {}'\mathfrak{D}_{t(\mu)}$ and $T_{t_0(\mu)} = \partial^\mu f/\partial t_0^\mu \times S(x)$. Using these notations the theorems in §4 are written in the following way.

**Theorem 4′.** *The mappings* $T \longrightarrow T_{t_0}$ *and* $T \longrightarrow \partial^\mu T/\partial t_0^\mu$ *are continuous.*

**Theorem 5′.** *If* $T \in {}'\mathfrak{D}_{t(\mu)}$ *then* $\partial^\lambda T/\partial t_0^\lambda \in {}'\mathfrak{D}_{t_0(\mu-\lambda)}$ *and* $\partial^\mu T/\partial t_0^\mu = \partial^{\mu-\lambda}(\partial^\lambda T/\partial t^\lambda)/\partial t_0^{\mu-\lambda}$ *for any* $0 \leq \lambda \leq \mu$.

**Theorem 7′.** *If* $T \in {}'\widetilde{\mathfrak{D}}_{t(\mu)}$, *then the mapping* $t_0 \longrightarrow \partial^\mu T/\partial t_0^\mu$ *is continuous.*

**Theorem 8.** *If for any* $t \in \mathfrak{V}$ *there corresponds* $T_t \in \mathfrak{D}'(x)$ *such that the mapping* $t \longrightarrow T_t$ *is continuous, then we can define* $(n+1)$-*dimensional distribution* $\widetilde{T}$ *by* $\langle \widetilde{T}, \varphi(x, t) \rangle = \int_{\mathfrak{V}} \langle T_t, \varphi(x, t) \rangle_x \, dt$ *where* $\langle \ \rangle_x$ *means the scaler product between* $\mathfrak{D}(x)$ *and* $\mathfrak{D}'(x)$. *Moreover* $\widetilde{T} \in {}'\widetilde{\mathfrak{D}}_{\mathfrak{V}}$ *for any* $\nu \geq 0$, *and* $\widetilde{T}_t = T_t$.

**Theorem 9.** *If a parametric distribution* $T_t$ *is* $\mu$-*times continuously differentiable with respect to* $t$ *on* $\mathfrak{V}$ *then* $\widetilde{T}$ *which is defined in Theorem 8 belongs to the space* $\bigcap_{\rho=0}^{\mu} {}'\widetilde{\mathfrak{D}}_{\mathfrak{V}(\rho)}$ *and* $\mu$-*times parametric derivative* $T_t^{(\mu)}$ *is equal to* $\partial^\mu \widetilde{T}/\partial t_0^\mu$ *or* $(\partial^\mu \widetilde{T}/\partial t^\mu)_{t_0}$ *on* $\mathfrak{V}$.

**Theorem 10.** *If* $T \in {}'\widetilde{\mathfrak{D}}_{\mathfrak{V}}$ *and* $\widetilde{T}$ *is constructed from* $T_t$ *on* $\mathfrak{V}$ *by Theorem 8, then* $\widetilde{T} = T$ *on* $\mathfrak{V}$.

**Theorem 11.** (*The converse of Theorem 9*). *If* $T \in {}'\widetilde{\mathfrak{D}}_{\mathfrak{V}} \cap \bigcap_{\rho}^{\mu}{}_{=1} {}'\mathfrak{D}_{\mathfrak{V}(\nu=\rho+1)}^{(\rho)}$ *then the mapping* $t \longrightarrow T_t$ *is* $\mu$-*times continuously differentiable from* $\mathfrak{V}$ *to* $\mathfrak{D}'(x)$, *and* $\mu$-*th parametric derivative* $T^{(\mu)}$ *equals to* $\partial^\mu T/\partial t_0^\mu$.

**Remark.** We have assumed $\nu=\rho+1$ in the space $'\widetilde{\mathfrak{D}}_{\mathfrak{B}(\rho)}$ in this theorem. This condition can be weakened, but it will not be sufficient to assume $\nu=\rho$ since $(\tau_{-h}\delta-\delta)/h \xrightarrow{(2)'} \delta'$, but not $\xrightarrow{(1)'} \delta'$.

6. We consider the following equation of evolution,

$$\partial U(x, t)/\partial t+\sum_{|\rho|\leq m} A_{\rho}(t) D_x^{\rho} U(x, t)=B(x, t). \qquad (1)$$

L. Schwartz treated the equation of this type,[3] where he considered the parametric distribution and the parametric equation of evolution. Now for example let $A_\rho(t) \in \mathfrak{E}(t)$ and let $B(x, t)$ be (as a mapping from $\mathfrak{B}$ to $\mathfrak{D}'(x)$) a continuous parametric distribution. $D_x^\rho$ means a differential operator in the space $\mathfrak{D}'(x)$. Moreover $B, A, U$ are all matricielle. In such a case he discussed the parametric continuously differentiable solution $U(x, t)$. Here we consider the proper distributional (in $\mathfrak{D}'(x, t)$) equation of this type and its proper distributional solution. (Initial condition on $t=t_0$ is given in the space $'\mathfrak{D}_{t_0}$.)

**Theorem 12.** *If a parametric continuously differentiable distribution $U(x, t)$ satisfies parametric equation* (1) *under the above-mentioned conditions, then $\widetilde{U}(x, t)$ satisfies the corresponding proper distributional equation,* i.e.

$$\partial \widetilde{U}(x, t)/\partial t+\sum_{|\rho|\leq m} A_{\rho}^{(t)} D_x^{\rho} \widetilde{U}(x, t)=\widetilde{B}(x, t).$$

**Theorem 13.** *If proper equation* (1) *is given, and proper solution $U(x, t)$ belongs to $'\mathfrak{D}_{\mathfrak{B}} \cap {}'\widetilde{\mathfrak{D}}_{\mathfrak{B}'(\nu=2)}$, then $U_t(x)$ satisfies the corresponding parametric equation.*

## References

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3) L. Schwartz: Ann. Inst. Fourier, **2**, 19–49 (1950).