73. A Note on Countably Paracompact Spaces

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In a recent note, the author[1)] proved that every normal space is countable collectionwise normal, i.e. these two concepts are equivalent. In this note, it is shown that, for normal spaces, countable paracompactness is equivalent to a property of topological importance.

Let X be a topological space. An open covering of X is called *countable*, if it is countable collection of open sets. The space X is called *countably paracompact*, if every countable open covering of X has a locally finite open refinement.

C. H. Dowker[2)] proved that the following conditions of a normal space are equivalent:

(1) The space X is countably paracompact.

(2) Every countable open covering of X has a point finite open refinement.

(3) Every countable open covering $\alpha=\{U_i\}$ of X has an open refinement $\beta=\{V_i\}$ such that $\overline{V}_i\subset U_i$ $(i=1, 2, \ldots)$.

We shall prove the following

Theorem. *For a normal space X, X is countably paracompact, if and only if, every countable open covering of X has a star-finite open refinement.*

Proof. Sufficiency follows immediately from the definition of countable paracompactness.

To prove the necessity, we take a countable open covering $\alpha=\{U_i\}$ of X. By C. H. Dowker's results, we can take a locally finite refinement $\beta=\{V_i\}$ of α such that $V_i\subset U_i$, and further there is a refinement $\gamma=\{W_i\}$ of β with $\overline{W}_i\subset V_i$ $(i=1, 2, \ldots)$. Let $V'_n=\bigcup_{i=1}^{n} V_i$, $W'_n=\bigcup_{i=1}^{n} W_i$, then $\overline{W'_n}\subset V'_n$. By the normality of X, for each pair V'_n, W'_n, there is a sequence of open sets V_n^j $(j=1,2,\ldots)$ such that

$$\overline{W'_n}\subset V_n^j\subset \overline{V_n^j}\subset V'_n,\quad \overline{V_n^j}\subset V_n^{j+1}\ (j=1, 2, \ldots).$$

We define G_i by

$$G_1=V_1^1,\quad G_2=V_2^1\cup V_1^2,\quad G_3=V_3^1\cup V_2^2\cup V_1^3, \ldots$$
$$\ldots G_n=\bigcup_{i+j=n+1} V_i^j, \ldots .$$

It is clear that each G_i is open in X, $\overline{G}_i\subset G_{i+1}\subset V'_i$ and $\bigcup_{i=1}^{\infty} G_i = X$. Following O. Hanner's argument,[3)] we construct O_i as follows:

$$O_1=G_1, \quad O_2=G_2, \quad O_n=G_n-\overline{G}_{n-2} \quad (n\geq 3).$$

Then each G_i is open in X, $O_n\subset G_n$, and since $\overline{G}_n\subset G_{n+1}$, $O_n\supset G_n-G_{n-1}$. Also $O_n\subset V_n'(n=1, 2, \ldots)$. Let $\delta=\{O_n\cap V_i \mid n=1, 2, \ldots i=1, 2, \ldots n\}$, then δ is a star-finite and a refinement of α. Thus the proof is complete.

References

1) K. Iséki: A note on normal spaces, to be appeared in Math. Japonicae.

2) C. H. Dowker: On countably paracompact spaces, Canadian Jour. Math., **3**, 219–224 (1951).

3) O. Hanner: Some theorems on absolute neighborhood retracts, Ark. Mat., **1**, 389–408 (1951).