95. On Hannerisation of Two Countably Paracompact Normal Spaces

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In this note, we shall prove the following

Theorem 1. The Hannerisation of two countably paracompact normal spaces is countably paracompact normal.

A space X is called *countably paracompact*, if every countable open covering of X has a locally finite open refinement. For normal space, such a space X can be characterized by the following condition: *every countable open covering of X has a star finite open refinement.* For the proof, see K. Iséki (3).

Let X and Y be normal spaces, B a closed subset of Y and $f:B\to X$ a mapping (continuous). Let $X\cup Y$ be the free union of X and Y, and Z the identification space obtained from $X\cup Y$ by identifying $x\in B$ with $f(x)\in X$. The natural mapping of $X\cup Y$ onto Z induces two mappings $j:X\to Z$ and $k:Y\to Z$. That is to say a subset O of Z is open if, and only if, $j^{-1}(O)$ and $k^{-1}(O)$ are open. Such a Z is called the Hannerisation of X and Y. It is well known that X is closed in Z and the partial mapping $k/Y-B$ is a homeomorphism onto $Z-X$.

O. Hanner [(1), (2)] proved that, if X and Y are both normal (resp. collectionwise normal, paracompact), then so is Z. E. Michael (5) observed that a similar result for perfectly normal space holds true. The present author (4) proved that, if X and Y are completely normal spaces, then so is Z.

Proof of Theorem 1. It is clear that Z is normal. Let $\alpha=\{O_n\}$ be any countable open covering of Z, then we shall show that α has a locally finite open refinement. The open covering $\{O_n\cap X\}$ of X has a star finite open refinement $\{U_n\}$, since X is countably paracompact normal. We can take O_{i_n} such that $U_n\subset O_{i_n}$ for each U_n. By a theorem of O. Hanner [(2), Lemma 7.2], there is a locally finite open covering $\{W_n\}$ of Z such that $U_n=W_n\cap X$. We can suppose that $W_n\subset O_{i_n}$ replacing W_n by $W_n\cap O_{i_n}$. If $Z=\bigcup_{n=1}^{\infty}W_n$, Z is countably paracompact, and if it is not, Hanner method [(2), p. 330] is available for our proof. Let $W=\bigcup_{n=1}^{\infty}W_n$, then W is an open set in Z such that $W\supset X$. Thus $k^{-1}(W)$ is open in Y, and $k^{-1}(W)\supset Z$. By the normality of Y, there is an open set V in Y

such that

$$Y-B\supset \overline{V}\supset V\supset Y-k^{-1}(W).$$

Since $\overline{V}$ is closed in Y, $\overline{V}$ is countably paracompact. Hence the open convering $\{k^{-1}(O_n)\cap \overline{V}\}$ of $\overline{V}$ has a locally finite open refinement $\{V_n\}$, where each V_n is open in $\overline{V}$. Thus $V_n\cap V$ is open in $Y-B$. Let $G_n=k(V_n\frown V)$, then G_n is open in Z, since $k/Y-B$ is a homeomorphism. On the other hand, since V is closed in Z, and $\{V_n\frown V\}$ is a locally finite in V, $k(V)$ is closed in Z and $\beta=\{G_n\}$ is a locally finite in $k(\overline{V})$. Hence $\beta=\{G_n\}$ is a locally finite in Z. $\bigcup_{n=1}^{\infty} G_n$ contains $Z-W$, for

$$\bigcup_{n=1}^{\infty}G_n=\bigcup_{n=1}^{\infty}k(V_n\frown V)=k(\bigcup_{n=1}^{\infty}(V_n\frown V))=k(V)\supset k(Y-k^{-1}(W))=Z-W.$$

Let $\gamma=\{G_n, W_n | n=1, 2,\dots\}$, then γ is a locally finite open covering of Z. It is obvious from the definition of γ that γ is a refinement of α. Hence Z is countably paracompact normal, and the proof is complete.

By the Theorem 1 and Hanner general method, we have the following

Theorem 2. If X is an absolute neighborhood retract with respect to the countably paracompact normal class Q, it is a neighborhood extension space for the Q.

Theorem 3. If X is an absolute retract with respect to the class Q (see Theorem 2), *it is an extension space for the Q.*

The converses of the Theorems 2, 3 hold true.

References

1) O. Hanner: Solid spaces and absolute retract, Ark. Mat., **1**, 375-382 (1951).

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4) K. Iséki: On Hannerisation of completely normal spaces, to appear in Revista da Faculdade de Ciencias de Lisboa.

5) E. Michael: Some extension theorems for continuous functions, Pacific J. Math., **3**, 789-806 (1953).