93. *Note on Linear Topological Spaces*

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§1. The purpose of this note is to give a generalization of Y. Kawada's theorem to convex linear topological spaces and some related remarks. As we treat only separative convex linear topological spaces, we shall call them merely *convex spaces*.

In the sequel, the word "Isomorphism" means always algebraic isomorphism together with homeomorphism, unless the contrary is mentioned. For two convex spaces E and F, $L(E, F)$ is the space of all continuous linear mappings of E to F. For any subset A of E and B of F, $(A \mid B)$ denotes the set; $\{u;\ u \in L(E, F)\ \ u(A) \subseteq B\}$. For a family $\mathfrak{S}$ of bounded subsets of E such that for any $A_1 \in \mathfrak{S}$ and $A_2 \in \mathfrak{S}$, there exists an $A_3 \in \mathfrak{S}$ with $A_1 \cup A_2 \subseteq A_3$ and $\bigcap_{A \in \mathfrak{S}} A = E$, we can define a convex linear topology in $L(E, F)$ whose basis of neighborhood of the origin consists of all $(A \mid V)$ where $A \in \mathfrak{S}$ and V is a neighborhood of o in F. This topology is called $\mathfrak{S}$-*topology*. We write $\langle x, x' \rangle$ instead of $x'(x)$ $(x \in E,\ x' \in E')$ where E' denotes the conjugate space of E. A neighborhood of the origin is called an *o-neighborhood*.

§2. Theorem 1. (Kawada[1]) *Let E and F be two convex spaces. If $L(E, E)$ and $L(F, F)$ are algebraically (ring) isomorphic, then there exists an algebraic isomorphic mapping φ of E onto F and $\tilde{\varphi}$ of E' onto F' such that* $\langle x, x' \rangle = \langle \varphi(x), \tilde{\varphi}(x') \rangle$ $(x \in E,\ x' \in E')$.

Proof. We sketch Kawada's proof.

(a) Any minimal left ideal $\mathfrak{A}$ of $L(E, E)$ is algebraically isomorphic to E in the manner $E \ni x \longleftrightarrow u_x \in \mathfrak{A}$ implies $v(x) \longleftrightarrow v \cdot u_x$ $(v \in L(E, E))$.

In fact, there exists an element x_0 of E and $u_0 \in \mathfrak{A}$ with $u_0(x_0) \neq 0$. We can easily see that the linear mapping $u \to u(x_0)$ maps $\mathfrak{A}$ onto E. The set $\{u;\ u \in \mathfrak{A}\ \ u(x_0) = 0\}$ is a left ideal contained in $\mathfrak{A}$ and not identical to it, so a zero ideal, because of the minimalness of $\mathfrak{A}$. Thus this mapping is an expected algebraic isomorphism. Conversely, the set $\{u_y;\ u_y(x) = \langle x, x_0' \rangle y,\ y \in E\}$ for non-zero $x_0' \in E'$ is a minimal left ideal.

(b) Let Φ be the given algebraic isomorphic mapping of $L(E, E)$ onto $L(F, F)$. Then

$$E \ni x \longleftrightarrow u_x \in \mathfrak{A} \longleftrightarrow \tilde{u}_{\tilde{x}} \in \tilde{\mathfrak{A}} \longleftrightarrow \tilde{x} \in F$$

and

$$v(x) \longleftrightarrow v \cdot u_x \longleftrightarrow \Phi(v)\tilde{u}_{\tilde{x}} \longleftrightarrow \Phi(v)[\tilde{x}]$$

1) Y. Kawada: "Ueber den Operatorenring Banachscher Räume", Proc. Imp. Acad., **19**, 616-621 (1943).

where $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are some corresponding minimal left ideals. Putting $\varphi(x)=\tilde{x}$, we get $\varphi(v(x))=\Phi(v)[\varphi(x)]$.

(c) For any $x' \in E'$, we put $u(x)=\langle x, x'\rangle x_0$ where x_0 is a fixed non-zero element of E. Then $\Phi(u)[z]=<\varphi^{-1}(z), x'>\varphi(x_0)$ $(z \in F)$. $\langle\varphi^{-1}(z), x'\rangle$ is a continuous linear functional on F, so may be written as a form $\langle z, \tilde{\varphi}(x')\rangle$ for some $\tilde{\varphi}(x') \in F'$. $\tilde{\varphi}$ gives a desired algebraic isomorphism. Q.E.D.

Corollary. (Mackey[2]) *If E and F are both relatively strong (or weak) in the sense of Mackey, then algebraic isomorphism of $L(E, E)$, and $L(F, F)$ implies isomorphism of E and F.*

When we consider topologies of E and F, we need the following lemma.

Lemma: *If $(C|D)$ is a $\mathfrak{S}$-o-neighborhood of $L(E, F)$ where D is a symmetric convex subset of F and C is neither empty nor reduced to the origin of E, then D is an o-neighborhood of F.*

Proof. By definition, there exists a symmetric convex closed bounded subset B of E and an o-neighborhood V of F such that $(B\,|\,V)\subseteq(C\,|\,D)$. For some non-zero real number α, C is not contained in αB. Assume that αV be not contained in D. Then we can find x_0, $\tilde{y}_0$ such that

$$C \ni x_0 \quad \text{and} \quad x_0 \bar{\in} \alpha B$$

and

$$\alpha V \ni \tilde{y}_0 \quad \text{and} \quad \tilde{y}_0 \bar{\in} D.$$

Since αB is symmetric, convex and closed, an application of Hahn-Banach theorem shows that there exists an $x_0' \in E'$ with the property,

$$\langle x_0, x_0'\rangle > 1 \quad \text{and} \quad |\langle x, x_0'\rangle| \leq 1 \ (x \in \alpha B).$$

The continuous linear mapping $u(x)=\langle x, x_0'\rangle\tilde{y}_0$, is contained in $(\alpha B\,|\,\alpha V)=(B\,|\,V)$, but not in $(C\,|\,D)$. This contradicts the hypothesis.

Theorem 2. *Suppose that the topologies of $L(E, E)$ and $L(F, F)$ are $\mathfrak{S}_E$- and $\mathfrak{S}_F$- topologies respectively. If $L(E, E)$ and $L(F, F)$ are isomorphic, then E and F are isomorphic.*

Proof: Let V be any o-neighborhood of E. From the proof of theorem 1, $u \in (\{x_0\}|V)$ if and only if $\Phi(u) \in (\{\varphi(x_0)\}|\varphi(V))$. Since $(\{x_0\}|V)$ is an o-neighborhood of $L(E, E)$ with respect to the given topology, $(\{\varphi(x_0)\}|\varphi(V))$ is an o-neighborhood of $L(F, F)$. By the lemma, $\varphi(V)$ is an o-neighborhood of F.

Remark. In theorem 2, the assumption of "homeomorphism" can not be omitted. For example, if E is a normed space with its norm topology and $F=E$ with its weak topology, then $L(E, E)$ and $L(F, F)$ are algebraically isomorphic but in general weak and norm topologies do not coincide.

2) G. W. Mackey: "On convex linear topological spaces", Trans. Amer. Math. Soc., **60**, 519-537 (1946).