121. On a Property of Mappings of Metric Spaces

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In his paper, "Solid Spaces and Absolute Retracts" (Ark. Mat., **1**, 375–382 (1952)), O. Hanner has proved that any metric NES (normal) is an absolute G_δ.

In this note, we shall prove the following theorem:

Any metric NES (completely normal) is an absolute G_δ.

Let α be a class of topological spaces, and A a space of α. The space Y is called an NES (α), if every mapping f of a closed subset A of a space X of α into Y can be extended to a mapping f' of an open set U into Y such that $A \subset U \subset X$.

A space X is called an absolute G_δ, if whenever X is topologically imbedded in a metric space Y, then X is a G_δ in Y.

To prove the theorem, we shall use the method employed by O. Hanner.

Let X_1 be any metric space containing X, and Z one to one with X_1. Let h be the $(1-1)$-correspondence from Z onto X_1. We shall introduce a topology in Z by taking as open sets in Z

$$h^{-1}(O) \cup A$$

where O is any open set of X_1 and A is any set of $Z-h^{-1}(X)$. Then h is continuous and $X'=h^{-1}(X)$ is closed in Z.

The topological space Z is completely normal. Let A_1, A_2 be separated sets in Z. Let $B_i=h(A_i \cap X')$ $(i=1, 2)$, then B_1, B_2 are separated in the metric space X_1. Therefore, the two sets $O_1=\{x | \rho(B_1, x)<\rho(B_2, x) \ \& \ x \in X_1\}$, $O_2=\{x|\rho(B_1, x)>\rho(B_2, x) \ \& \ x \in X_2\}$ are disjoint open. Hence $U_1=h^{-1}(O_1) \cup A_1$, $U_2=h^{-1}(O_2) \cup A_2$ are disjoint open in Z, and $U_i \supset A_i$ $(i=1, 2)$. Thus Z is completely normal.

To prove that X is an absolute G_δ, we shall use an argument of C. H. Dowker.[1]

The partial mapping $h | X' \rightarrow X$ is extended to a mapping h' of an open set U, such that $X' \subset U \subset X$, into X, since X is an NES (completely normal). Let

$$f(x)=\rho(h(x), h_1(x)) \qquad \text{for} \qquad x \in U,$$

then $f(x)$ is continuous, and $f(x)=0$ if and only if $x \in X'$. This shows that X' is a G_δ in U. There are open sets U_n in U such that $X' = \bigcap_{n=1}^{\infty} U_n$. Hence every U_n is open in Z, and $h(U_n)=V_n \cup A_n$ $(n=1, 2, \cdots)$ where V_n is open in X_1 and $A_n \subset X_1 - X$. Thus $X=\bigcap_{n=1}^{\infty} V_n$ and X

is a G_δ in X_1. This completes the proof.

In a recent note,[2] the present author introduced a notion, absolute neighborhood retract for the class of completely normal spaces and proved that X is ANR (completely normal) if and only if it is NES (completely normal). Therefore, we have the following

Theorem. Any metric ANR (completely normal) is an absolute G_δ, and hence topologically complete.

By my previous results[3] and similar method, we have the following

Theorem. Any metric ANR (countably paracompact) is an absolute G_δ.

The theorem for paracompact spaces was proved by O. Hanner ((4), p. 333, Theorem 14.1).

Theorem. Any separable metric ANR (hypocompact) is an absolute G_δ.

This is proved by a similar argument using a result of S. Kaplan ((5), p. 249).

References

1) C. H. Dowker: On a theorem of Hanner, Ark. Mat., **2**, 307–313 (1952).

2) K. Iséki: On the Hannerisation of completely normal spaces, to appear soon.

3) K. Iséki: On Hannerisation of two, countably paracompact normal spaces, Proc. Japan Acad., **30**, 443–444 (1954).

4) O. Hanner: Retraction and extension of mappings of metric and non-metric spaces, Ark. Mat., **2**, 315–360 (1952).

5) S. Kaplan: Homology theory of arbitrary subsets of Euclidean spaces, Trans. Am. Math. Soc., **62**, 248–271 (1947).