177. A Characterization of Hilbert Space

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It is our purpose in this note to prove the following

THEOREM. *A Banach space E is unitary if and only if it satisfies the condition.*

(*) *There is assigned to E a positive number α not greater than 1/2, and for any x, y in E, there exists at least a λ, $\alpha \leqq \lambda \leqq 1-\alpha$, which depends on x and y, such that*

$$\lambda \| x \|^2 + (1-\lambda) \| y \|^2 \geqq \lambda(1-\lambda) \| x-y \|^2 + \| \lambda x + (1-\lambda) y \|^2,$$

where $\| \ \|$ is the norm.

Whenever we speak of a Banach space we shall mean a Banach space over real field R.

We shall only prove the "if" part of the theorem since the "only if" part is clear. Using Kakutani's result,[1] it is sufficient to show that for any closed linear subspace M of E, there exists an extension of the identity transformation of M which is linear continuous and has norm 1. From the fact that the continuous linear map of a linear subspace N of a Banach space into another Banach space F can be extended to a continuous linear map of the closure $\bar{N}$ into F without changing the norm, and by virtue of Zorn's lemma, our problem can be simplified in the form: to prove the following statement.

Let E be a Banach space satisfying the condition (*), and M a closed hyperplane. Then the identity transformation I of M can be extended to a continuous linear transformation of E onto M whose norm is 1.

For this purpose, we shall need the lemmata below.

LEMMA 1. *Let E be a Banach space satisfying the condition (*). If $x, y \in E$ are such that:*

$$\max [\| x \|, \| y \|] < \| x-y \|$$

then there is a λ, $0 < \lambda \leqq 1-\alpha$, which insures

$$\| \lambda x + (1-\lambda) y \| < \min [\| x \|, \| y \|].$$

Proof. We may suppose $\| y \|$ is not greater than $\| x \|$. In 2-dimensional Euclidean space, we constract a triangle with verteces

1) S. Kakutani: *Some characterizations of Euclidean space*, Japanese Jour. Math., **16**, 93–97 (1939).

O, X and Y such that

$$|X-O|=\|x\|,\ |Y-O|=\|y\|,\ |X-Y|=\|x-y\|.$$

For any λ, $0<\lambda<1$, let us denote by $Z(\lambda)$ a point on the segment $\overline{XY}$ for which $|Z(\lambda)-Y|=\lambda|X-Y|$.

Now by the assumption, $|X-O|$ is smaller than $|X-Y|$ and not smaller than $|Y-O|$, so there is a λ_0, $0<\lambda_0<1$, such that $|Z(\lambda)-O|<|Y-O|$ whenever $0<\lambda<\lambda_0$.

On the other hand, we have from brief calculations

$$|Z(\lambda)-O|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2;$$

and the condition (*) insures the existence of a λ_1, $\alpha\leqq\lambda_1\leqq 1-\alpha$, such that

$$\lambda_1\|x\|^2+(1-\lambda_1)\|y\|^2-\lambda_1(1-\lambda_1)\|x-y\|^2\geqq\|\lambda_1x+(1-\lambda_1)y\|^2.$$

We shall consider the case where λ_1 gives the inequality. Now if the inequality holds true for all λ with $0<\lambda<1-\alpha$, then the lemma is clear, because we can choose a λ smaller than λ_0, and hence $\|\lambda x+(1-\lambda)y\|<\|y\|$ for this λ. Otherwise, there is a λ, $0<\lambda\leqq 1-\alpha$, such that

$$(=)\qquad \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2=\|\lambda x+(1-\lambda)y\|^2$$

since the norm is continuous.

Thus it will suffice to prove the lemma under following condition on the norm.

(**) For any x, $y\in E$, there exists a λ, $0<\lambda\leqq 1-\alpha$, such that the equality (=) holds.

Therefore, we may assume that the equality (=) holds for λ_1; if λ_1 is not smaller than λ_0, we consider the triangle $XYZ(\lambda_1)$. Then we can take a λ_2, $0<\lambda_2\leqq(1-\alpha)^2$, for which (=) is valid in view of condition (**). Further, if $\lambda_2\geqq\lambda_0$, we consider the triangle $XYZ(\lambda_2)$, and so on. Since $(1-\alpha)^n$ tends to zero as $n\to\infty$, we have $\lambda_n<\lambda_0$ for sufficiently large n, proving the lemma.

LEMMA 2. *A closed convex set C in a Banach space satisfying the condition (*) contains a unique element of smallest norm.*

Proof. Let $\rho=\inf_{x\in C}\|x\|$ and choose $x_m\in C$ that may satisfy $\|x_n\|\downarrow\rho$. Then for any $\varepsilon>0$, there is an integer N such that

$$\|x_n\|<\rho+\varepsilon,\quad \|x_m\|<\rho+\varepsilon,$$

for any m, $n>N$.

The condition (*) insures the existence of a λ, $\alpha\leqq\lambda\leqq 1-\alpha$, such that

$$\|x_n-x_m\|^2\leqq\frac{1}{1-\lambda}\|x_n\|^2+\frac{1}{\lambda}\|x_m\|^2 -\frac{1}{\lambda(1-\lambda)}\|\lambda x_n+(1-\lambda)x_m\|^2.$$

Now since C is convex, $\lambda x_n+(1-\lambda)x_m$ is in C, so that

$$\|x_n-x_m\|^2<\frac{(\rho+\varepsilon)^2}{\lambda(1-\lambda)}-\frac{\rho^2}{\lambda(1-\lambda)}<\frac{(2\rho+\varepsilon)\varepsilon}{\alpha^2}.$$

Let $x_0=\lim_{n\to\infty} x_n$, then x_0 is in C since C is closed, and it follows from the continuity of the norm that $\|x_0\|=\rho$. It is an immediate consequence of Lemma 1 and the condition $(*)$ that the element x_0 is unique.

We shall now proceed to prove the above-mentioned statement. Let x_0 be an element of E which does not belong to M; then the set $\{y-x_0 \mid y\in M\}$ is clearly convex and closed, so by Lemma 2 there is a unique element y_0 such that $\|y_0-x_0\|\leqq\|y-x_0\|$ for all $y\in M$.

It is easy to see that for all $y\in M$, we have

$$\|y-y_0\|\leqq\|y-x_0\|.$$

In fact, if $\|y-y_0\|$ is greater than $\|y-x_0\|$ for some $y\in M$, then in virtue of Lemma 1 there exists a λ, $0<\lambda<1-\alpha$, such that

$$\|\lambda y+(1-\lambda)y_0-x_0\|<\|y_0-x_0\|$$

which is a contradiction since $\lambda y+(1-\lambda)y_0$ is in M.

Now we define $\quad I^*(x)=I(y)+\lambda y_0=y+\lambda y_0$

for any $\quad x=y+\lambda x_0,\ y\in M,\ \lambda\in R$.

Then it is clear that I^* is linear and an extension of I to $M+Rx_0$, and hence it remains only to prove the continuity of I^* and that the norm is 1. For that matter the relation

$$\|y+\lambda y_0\|=|\lambda|\cdot\|\lambda^{-1}y+y_0\|$$

holds for $\lambda\neq 0$.

On the other hand, $\quad \|\lambda^{-1}y+y_0\|\leqq\|-\lambda^{-1}y-x_0\|,$

and so $\quad \|y+\lambda y_0\|\leqq\|y+\lambda x_0\|,$

which guarantees the continuity of I^* and shows the norm is 1. Thus we have reached the desired conclusion.

Additions and Corrections to Shouro Kasahara: "A Note on f-completeness"

(Proc. Japan Acad., **30**, No. 7, 572–575 (1954))

Pages 572–573, delete "Proposition 2".

Page 574, delete "Proposition 6".

Page 574, line 19 from foot, for "mapping of W, we have $p(I^*(x))\leqq p(x)$ for any $p\in(p_\alpha)$ and $x\in E$." read "mapping of W, concerning to $p\in(p_\alpha)$, we have $p^*(I(x))\leqq p(x)$ for any $x\in E$.".

Page 574, lines 26–29, delete "Now, since$\cdots$inequality $(*)$ for u^*.".

Pape 574, line 10 from foot, for "for any $p\in(p_\alpha)$ there is" read "there exist a $p\in(p_\alpha)$ and".

Page 574, line 2 from foot, for "*same a*" read "*same p and a*".