174. Dirichlet Problem on Riemann Surfaces. III
(Types of Covering Surfaces)

By Zenjiro KURAMOCHI
Mathematical Institute, Osaka University

(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1954)

Let $\underline{R}$ be a null-boundary Riemann surface and let R be a positive boundary Riemann surface given as a covering surface.

1) If $\mu(R, \mathfrak{A}(R, \underline{R}^*))=1$, we call R a covering surface of D-type over $\underline{R}$.

2) We map R^{∞} onto the unit-circle $U_{\xi}: |\xi|<1$ conformally. If the composed function $\underline{z}=\underline{z}(\xi): U_{\xi}\to R\to \underline{R}^*$ has angular limits with respect to $\underline{R}$ almost everywhere on $|\xi|=1$. We call R a covering surface of F-type over $\underline{R}$.

3) Let $T(r)$ be the characteristic function of the mapping $R\to\underline{R}$. If $T(r)$ is bounded, we say, R is a covering surface of bounded type. By Theorem 1.1, it is easy to see that we have

Bounded type $\to$[1] F-type $\to$ D-type, and that F-type implies $\mu(R^{\infty}, \mathfrak{A}(R^{\infty}, \underline{R}^*))=1$. If the universal covering surface of the projection of R is hyperbolic, $\mu(R^{\infty}, \mathfrak{A}(R^{\infty}, \underline{R}^*))=1$ implies that R is a covering surface of F-type, because $\mu(R^{\infty}, \mathfrak{A}(R^{\infty}, B))$[2] $=0$.

Let $\hat{R}$ be a covering surface over R. In the following, we investigate the relations between Riemann surface $\hat{R}$ and R. By Theorem 1.1 we have at once the following

Theorem 3.1. If R is a covering surface of bounded type, then $\hat{R}$ is also of bounded type relative to $\underline{R}$.

Theorem 3.2. Let R be a covering surface such that the universal covering surface of the projection $\underline{R}'^{\infty}$ of R is hyperbolic. We map $\underline{R}'^{\infty}$, R^{∞} and $\hat{R}^{\infty}$ conformally onto the unit-circles $U_{\xi}: |\xi|<1$, $U_{\eta}: |\eta|<1$ and $U_{\zeta}: |\zeta|<1$ respectively. Let $\eta=\eta(\zeta)$, $\xi=\xi(\zeta)$ and $\xi=\xi(\eta)$ be mappings $U_{\zeta}\to U_{\eta}$, $U_{\zeta}\to U_{\xi}$ and $U_{\zeta}\to U_{\xi}$ respectively. Then we have

$$\mu(\hat{R}^{\infty}, \mathfrak{A}(\hat{R}^{\infty}, \underline{R}^*)) \geqq \mu(R^{\infty}, \mathfrak{A}(R^{\infty}, \underline{R}^*)).$$

Proof. Since $\mu(\underline{R}'^{\infty}, \mathfrak{A}(\underline{R}'^{\infty}, B))=\mu(R^{\infty}, \mathfrak{A}(R^{\infty}, B))=\mu(\hat{R}^{\infty}, \mathfrak{A}(\hat{R}^{\infty}, B))=0$ without loss of generality, we can suppose that every A.B.P. lies on $\underline{R}$. Let A_{η} and A_{ζ} be images of $\mathfrak{A}(R^{\infty}, \underline{R})$ and $\mathfrak{A}(\hat{R}^{\infty}, \underline{R})$ respectively, and let ${}_{\eta}S_{\zeta}$, ${}_{\xi}S_{\zeta}$ and ${}_{\xi}S_{\eta}$ be the sets where the corresponding functions

---

1) $\to$ means implication.

2) Measure of a set of A.B.P.'s of R^{∞} with projections on the ideal boundary B of $\underline{R}$.

have angular limits on $\overline{U}_\eta: |\eta| \leqq 1$, $\overline{U}_\zeta: |\zeta| \leqq 1$ and $\overline{U}_\eta: |\eta| \leqq 1$ respectively. Then $\mathrm{mes}\,{}_\eta S_\zeta = \mathrm{mes}\,{}_\xi S_\zeta = \mathrm{mes}\,{}_\xi S_\eta = 2\pi$. Take a point $\zeta_0 \in ({}_\xi S_\zeta \cap {}_\eta S_\zeta \cap CA_\zeta)$ and let l_{ζ_0} be the radius terminating at ζ_0, where CA_ζ is the complementary set of A_ζ with respect to the circumference of U_ζ. If l_η, the projection of l_{ζ_0} on U_η, tends to a point $\eta_0: |\eta_0| < 1$, l_η determines an A.B.P., whence $\zeta_0 \in A_\zeta$. This is absurd. Next, assume that l_η converges to an arc γ on $|\eta| = 1$ such that $\gamma \cap A_\eta \neq 0$. Take a point $\eta_0 \in A_\eta$ and let l' be the radius terminating at η_0. Then l_η intersets l' infinitely many times. It follows that l_η determines an A.B.P. angularly, because the image l_ξ on U_ξ of l_η and the image l'_ξ of l' tends to the same point ξ_0 in U_ξ. Thus $\zeta_0 \in A_\zeta$. Suppose l'_η intersects an angular domain $A_\eta(\theta)$: $|\arg(1-e^{-i\theta}\eta)| < \frac{\pi}{2} - \delta$, $e^{-i\theta} \in A_\eta$ infinitely many times, then we have also that $\zeta_0 \in A_\zeta$. Hence, if ζ tends in an angular domain $A_\zeta(\theta)$ at every point of $CA_\zeta \cap {}_\xi S_\zeta \cap {}_\eta S_\zeta$, $\eta = \eta(\zeta)$ tends to $CA_\eta + C_\xi S_\eta$ or tends to A_ζ tangentially. Let $F(\zeta)$ and $F(\eta)$ be closed subsets in $CA_\zeta \cap {}_\xi S_\zeta \cap {}_\eta S_\zeta$ and in A_η respectively, and let $D_\delta(F(\zeta))$ and $D_\delta(F(\eta))$ be domains such that $D_\delta(F(\zeta))$ and $D_\delta(F(\eta))$ contain angular endparts: $\arg|1-e^{-i\theta}\zeta| < \frac{\pi}{2} - \delta$, $e^{i\theta} \in F(\zeta)$ and $\arg|1-e^{-i\theta}\eta| < \frac{\pi}{2} - \delta$, $e^{i\theta} \in F(\eta)$ respectively and let $C'_r(\zeta)$ and $C'_r(\eta)$ be the rings such that $r < |\zeta| < 1$ and $r < |\eta| < 1\ (r < 1)$. From above consideration, since $\xi = \xi(\eta)$ has angular limits in U_ξ at every point of A_η. There exists a subset $A_{\eta,n}$ of A_η such that angular limits at $A_{\eta,n}$ are contained in $|\xi| < 1 - \frac{1}{n}$ and $\mathrm{mes}\,|A_\eta - A_{\eta,n}| < \frac{\varepsilon}{2}$. Therefore there exists a closed subset $F(\eta)$ of $A_{\eta,n}$ and r, for δ, such that $\mathrm{mes}\,|A_{\eta,n} - F(\eta)| < \frac{\varepsilon}{2}$ and if $\eta \in (D_\delta(F(\eta)) \cap C'_r(\eta))$, then $|\xi(\eta)| < 1 - \frac{1}{2n}$. On the other hand since $\xi = \xi(\zeta)$ has angular limits at every point $CA_\zeta \cap {}_\xi S_\zeta$ which lie on $|\xi| = 1$, there exist r' and a closed subset $F(\zeta)$ of CA_ζ such that $\mathrm{mes}\,|CA_\zeta - F(\zeta)| < \varepsilon$ and if $\zeta \in (D_\delta(F(\zeta)) \cap C'_{r'}(\zeta))$, then $\eta = \eta(\zeta) \notin D_\delta(F(\eta))$. Denote by $C_r(\eta)$ a circle such that $|\eta| < r\ (r < 1)$ and let $v(\eta)$ be a continuous super-harmonic function in U_η such that $v(\eta)$ is harmonic in $D_\delta(F(\eta)) \cup C_r(\eta)$, $v(\eta) = 1$ on the boundary of $(D_\delta(F(\eta)) \cup C_r(\eta))$ not lying on $|\eta| = 1$, $v(\eta) \equiv 1$ on $U_\eta - (D_\delta(F(\eta)) \cup C_r(\eta))$ and $v(\eta) = 0$ on the boundary of $((D_\delta(F(\eta)) \cup C_r(\eta))$ lying on $|\eta| = 1$. Consider $v(\eta)$ on $C_{r'}(\zeta) \cup D_\delta(F(\zeta))$, then $v(\zeta) = v(\eta)$ is a function such that $\lim v(\zeta) = 1$ when ζ tends to $F(\zeta)$. Since the boundary of $(C'_{r'}(\zeta) \cup D_\delta(F(\zeta)))$ is rectifiable and we can take δ arbitrarily, we have $\mu(U_\zeta, F(\zeta)) \leqq \mu(U_\eta, CF(\eta))$, where $\mu(U_\zeta, F(\zeta))$ and $\mu(U_\eta, CF(\eta))$ are the lower envelopes of $\{v(\zeta)\}$ which are the class of continuous super-harmonic functions in $D_\delta(F(\zeta))$ such that $0 \leqq v(\zeta) \leqq 1$ and $\lim v(\zeta) = 1$, when ζ tends to $F(\zeta)$ and of $\{v(\eta)\}$ respectively. Let $\varepsilon \to 0$. Then we have $\omega(U_\zeta, CA_\zeta) \leqq \omega(U_\eta, CA_\eta)$. Since A_ζ and A_η are measurable,

$$\mu(\widehat{R}^\infty, \mathfrak{A}(\widehat{R}^\infty, \underline{R}^*)) \geqq \mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*)).$$

Corollary. If the universal covering surface of the projection of R is hyperbolic and R is of F-type, then $\widehat{R}$ is also of F-type over $\underline{R}^*$, where $\widehat{R}$ is a covering surface over R.

If the universal covering surface of the projection $\underline{R}'$ of R is parabolic, remove a finite number of point $p_i\,(i=1,2,\dots n)$ so that $(\underline{R}' - \sum_{i=1}^{n} p_i)^\infty$ may be hyperbolic. Let $\widehat{R}$ be a covering surface R and let $p_{ij}\,(j=1,2,\dots)$ be points of R lying on p_i and $p_{ijk}\,(k=1,2,\dots)$ be points of $\widehat{R}$ lying on p_{ij}. Put $\widetilde{R} = R - \sum_{ij} p_{ij}$ and $\widetilde{\widehat{R}} = \widehat{R} - \sum_{ijk} p_{ijk}$. We map R^∞, $\widetilde{R}^\infty$, $\widehat{R}^\infty$ and $\widetilde{\widehat{R}}^\infty$ and $(\underline{R}' - \sum_{i=1}^{n} p_i)^\infty$ onto $U_\eta : |\eta| < 1$, $U_{\tilde{\eta}} : |\tilde{\eta}| < 1$, $U_\zeta : |\zeta| < 1$, $U_{\tilde{\zeta}} : |\tilde{\zeta}| < 1$ and $U_\xi : |\xi| < 1$ conformally respectively. Let $A_{\tilde{\eta}}$ and $A_{\tilde{\zeta}}$ be images of A.B.P.'s of $\widetilde{R}$ and $\widetilde{\widehat{R}}$.

Theorem 3.3. Let R be a positive boundary Riemann surface. If R covers p_i so few times that $\sum G(z, p_{ij}) < \infty$ and if

$$\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*)) = \mu(\widetilde{R}^\infty, \mathfrak{A}(\widetilde{R}^\infty, \underline{R}^*)) = \omega(U_{\tilde{\eta}}, A_{\tilde{\eta}}),$$

then for every covering surface $\widehat{R}$ over R,

$$\mu(\widehat{R}^\infty, \mathfrak{A}(\widehat{R}^\infty, \underline{R}^*)) = \mu(\widetilde{\widehat{R}}^\infty, \mathfrak{A}(\widetilde{\widehat{R}}^\infty, \underline{R}^*)) = \omega(U_{\tilde{\zeta}}, A_{\tilde{\zeta}}),$$

where $G(z, p_{ij})$ is the Green's function of R with pole at p_{ij}.

Proof. 1) As to $\widehat{R}^\infty$ and $\widetilde{\widehat{R}}^\infty$, let $\widehat{A}_i$ and $\widetilde{\widehat{A}}_i$ be the images of A.B.P.'s with projection on R of $\widehat{R}^\infty$ and $\widetilde{\widehat{R}}^\infty$ respectively. Then $\widehat{A}_i$ and $\widetilde{\widehat{A}}_i$ are Borel sets and $\eta = \eta(\zeta)$ and $\eta = \eta(\tilde{\zeta})$ have angular limits contained in U_η at every points of $\widehat{A}_i$ and $\widetilde{\widehat{A}}_i$. Let $\{\eta_{ijs}\}$ $(s=1,2,\dots)$ be images of p_{ij} in U_η and let $\{\zeta_{ijkt}\}$ $(t=1,2,\dots)$ be images of p_{ijk} in U_ζ. Since $\sum_{ijk} G(\hat{z}, p_{ijk}) \leqq \sum_{ij} G(z, p_{ij}) < \infty$, $\infty > \sum \log \left| \frac{1 - \eta\,\overline{\eta_{ijs}}}{\eta - \eta_{ijs}} \right| \geqq \sum \log \left| \frac{1 - \overline{\zeta}_{ijkt}\zeta}{\zeta - \zeta_{ijkt}} \right|$ and $\sum (1 - |\zeta_{ijkt}|) < \infty$, where $G(\hat{z}, p_{ijk})$ is the Green's function of $\widehat{R}$ with pole at p_{ijk}.

Let l and l' be paths in $\widehat{R}^\infty$ and $\widetilde{\widehat{R}}^\infty$ determining an A.B.P. not lying on p_{ij} and not lying on p_{ijk} respectively. Since we can deform l and l' as little as we please, we can suppose that the projection of l and l' do not pass p_{ij}.

2) Let $\widetilde{\widehat{A}}_i'$ be the image of A.B.P.'s of $\widetilde{\widehat{R}}^\infty$ whose projection lie

on p_{ij} of R. Since $\sum_{ij} G(z, p_{ij})<\infty$, $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, \sum p_{ij}))=0$. We consider only A.B.P.'s not lying on p_{ij}. Since $\widetilde{\widehat{R}}^{\infty}$ and $\widehat{R}^{\infty}$ are covering surfaces, we can consider $\widehat{A}_i$ and $\widetilde{\widehat{A}}_i$ the images of A.B.P.'s of $\widehat{R}^{\infty}$ and $\widetilde{\widehat{R}}^{\infty}$ lying in U_η. Hence $\widehat{A}_i$ and $\widetilde{\widehat{A}}_i$ are Borel sets. Since $\widetilde{\widehat{R}}^{\infty}$ is the universal covering surface of $(U_\zeta - \sum \zeta_{ijks})$,

$$\omega(U_\zeta, \widehat{A}_i)=\mu(\widehat{R}^{\infty}, \mathfrak{A}(\widehat{R}^{\infty}, R)) \geqq \mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))=\omega(U_{\tilde{\zeta}}, \widetilde{\widehat{A}}_i).$$

Since $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))$ is harmonic in $\widetilde{\widehat{R}}$, $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))$ is a single valued harmonic function in U_ζ. We denote by E_λ the set on $|\zeta|=1$ where $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))$ has angular limits $\lambda\,(\lambda<1)$. We show $\operatorname{mes}(\widehat{A}_i \cap E_\lambda)=0$. Denote the radial segments from ζ_{ijkt} to $|\zeta|=1$ by S_{ijkt} and put $(U_\zeta - \sum_{ijkt} S_{ijkt})=U'_\zeta$. Then U'_ζ is a simply connected domain with a rectifiable boundary. Consider the function $\zeta=\zeta(\tilde{\zeta})$. Then the inverse fuction $\tilde{\zeta}=\tilde{\zeta}(\zeta)$ is also single valued and U'_ζ is mapped into $U_{\tilde{\zeta}}$ conformally such that the image of U'_ζ covers $U_{\tilde{\zeta}}$ at most once. Let l_ζ be a radial path in U'_ζ terminating at $\widehat{A}_i$ and let $l_{\tilde{\zeta}}$ be the image in $U_{\tilde{\zeta}}$ of l_ζ. Then $l_{\tilde{\zeta}}$ is a path determining an A.B.P. lying on R. Hence $l_{\tilde{\zeta}}$ tends to a point in $\widetilde{\widehat{A}}_i$. Let $\widetilde{\widehat{A}}'_i$ be the set of points which is an endpoint of $l_{\tilde{\zeta}}$ above-mentioned. Then $\widetilde{\widehat{A}}'_i\,(\subset \widetilde{\widehat{A}}_i)$ is an analytic set. Since $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))$ has limit λ along l_ζ when ζ tends to $\widehat{A}_i \cap E_\lambda$, $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))$ has limit λ along the image $l_{\tilde{\zeta}}$ of l_ζ. Hence at every point of the image $\overset{\frown}{(\widehat{A}_i \cap E_\lambda)}$ of $(\widehat{A}_i \cap E_\lambda)$ $\mu(\widehat{R}^{\infty}, \mathfrak{A}(\widehat{R}^{\infty}, R))$ has angular limits smaller than 1. Since $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R)) = \omega(U_{\tilde{\zeta}}, \widetilde{A}_i)$, $\operatorname{mes}\overset{\frown}{(\widehat{A}_i \cap E_\lambda)}=0$. On the other hand, we map U'_ζ ont $|\zeta'|<1$. Then $|\zeta'|<1$ is a covering surface over $U_{\tilde{\zeta}}$, and $(\widehat{A}_i \cap E_\lambda)$ is transformed to a set $(\widehat{A}_i \cap E_\lambda)'$ on $|\zeta'|=1$. Then by Löwner's lemma, $\operatorname{mes}(\widehat{A}_i \cap E_\lambda)' \leqq \operatorname{mes}\overset{\frown}{(\widehat{A}_i \cap E_\lambda)}=0$. Since the boundary of U'_ζ is rectifiable, $\operatorname{mes}(\widehat{A}_i \cap E_\lambda)=0$. Hence $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))$ has angular limits 1 almost everywhere on $\widehat{A}_i$. Thus $\mu(\widehat{R}^{\infty}, \mathfrak{A}(\widehat{R}^{\infty}, R)) \leqq \mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))$ and $\mu(\widetilde{\widehat{R}}^{\infty}, \mathfrak{A}(\widetilde{\widehat{R}}^{\infty}, R))=\mu(\widehat{R}^{\infty}, \mathfrak{A}(\widehat{R}^{\infty}, R))$.

Consider $\mu(\widetilde{R}^{\infty}, \mathfrak{A}(\widetilde{R}^{\infty}, \underline{R}^*))$ on $\widetilde{\widehat{R}}^{\infty}$. Denote by $\widetilde{\widehat{A}}$ the set on $|\zeta|=1$ where at least one curve determining an A.B.P. terminates and by $C\widetilde{\widehat{A}}$ its complement. We show $\mu(\widetilde{R}^{\infty}, \mathfrak{A}(\widetilde{R}^{\infty}, \underline{R}^*))$ has angular limits 0 almost everywhere $C\widetilde{\widehat{A}}$. Assume there exists a set $\widetilde{\widehat{E}}_\delta$ of positive measure contained in $C\tilde{\tilde{A}}$ where $\mu(\tilde{R}^{\infty}, \mathfrak{A}(\tilde{R}^{\infty}, \underline{R}^*))$ has angular limits $\delta(\delta>0)$. Consider the mapping function $\xi=\xi(\tilde{\zeta})$, $\eta=\eta(\tilde{\zeta})$ and denote by ${}_{\xi}S_{\tilde{\zeta}}$ and by ${}_{\eta}S_{\tilde{\zeta}}$ the sets of point such that the corresponding functions $\xi=\xi(\tilde{\zeta})$ and $\eta=\eta(\tilde{\zeta})$ have angular limits on $|\xi|\leqq 1$ and $|\eta|\leqq 1$ respectively. On the other hand let $\tilde{A}^n_{\eta}$ be the set of $\tilde{A}_{\eta}$, images of A.B.P.'s of $\tilde{R}^{\infty}$ whose projection is contained in $|\xi|<1-\frac{1}{n}$. Then $\lim\limits_{n=\infty}|\operatorname{mes}(\tilde{A}_{\eta}-\tilde{A}^n_{\eta})|=0$. Let $l_{\tilde{\zeta}}$ be a Stolz's path terminating at $\tilde{\tilde{E}}_{\delta}$ and let $l_{\tilde{\eta}}$ be its image. Then we see $l_{\tilde{\zeta}}$ terminates at $A_{\tilde{\eta}}$ tangentially or $CA_{\tilde{\eta}}$ (Theorem 3.2). But since $\mu(\tilde{R}^{\infty}, \mathfrak{A}(\tilde{R}^{\infty}, \underline{R}^*))$ has limits δ along l_{η}, l_{η} does not tend to a point where $\mu(\tilde{R}^{\infty}, \mathfrak{A}(\tilde{R}^{\infty}, \underline{R}^*))$ has angular limits 0. Therefore l_{η} tends to the set $\tilde{E}_{\lambda}$ where $\mu(\tilde{R}^{\infty}, \mathfrak{A}(\tilde{R}^{\infty}, \underline{R}^*))$ has angular limits $\lambda(0<\lambda<1)$ or to the set where $\mu(\tilde{R}^{\infty}, \mathfrak{A}(\tilde{R}^{\infty}, \underline{R}^*))=1$ tangentially. Now since $\operatorname{mes}|E_{\lambda}\cap CA_{\tilde{\eta}}|=0$ and by Löwner's lemma, we have $\operatorname{mes}|\tilde{\tilde{E}}_{\delta}|=0$. Hence $\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*)) \geqq \mu(\tilde{R}^{\infty}, \mathfrak{A}(\tilde{R}^{\infty}, \underline{R}^*))$. Let $A^b_{\tilde{\zeta}}$ be the set on $|\zeta|=1$ where at least one curve determining an A.B.P. not lying on R. Then $A^b_{\tilde{\zeta}}$ is measurable and

$$\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*))=\omega(U_{\tilde{\zeta}}, \tilde{\tilde{A}}_t)+\omega(U_{\tilde{\zeta}}, A^b_{\tilde{\zeta}})\geqq \mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, R)).$$

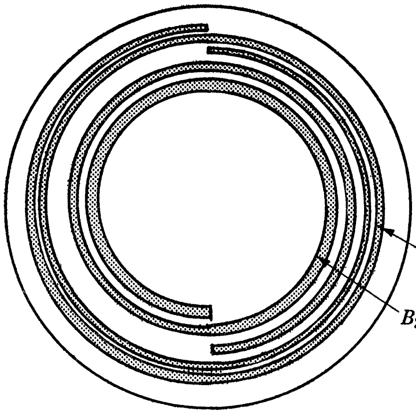
But $\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*))\geqq 0$ on $\tilde{\tilde{A}}_t$ where $\omega(U_{\tilde{\zeta}}, \tilde{\tilde{A}}_t)=1$ almost everywhere. Hence $\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*))$ has the same angular limits as $\operatorname{Min}[1, \mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*))+\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, R)]$. Since $\hat{R}^{\infty}$ is a covering surface over R^{∞}, $\mu(\hat{R}^{\infty}, \mathfrak{A}(\hat{R}^{\infty}, \underline{R}^*)\leqq \operatorname{Min}[1, \mu(R^{\infty}, \mathfrak{A}(R^{\infty}, \underline{R}^*))+\mu(\hat{R}^{\infty}, \mathfrak{A}(\hat{R}^{\infty}, R))]$. On the other hand by assumption $\mu(\tilde{R}^{\infty}, \mathfrak{A}(\tilde{R}^{\infty}, \underline{R}^*))$ $\mathfrak{A}=\mu(R^{\infty}, \mathfrak{A}(R^{\infty}, \underline{R}^*))=\omega(U_{\tilde{\zeta}}, A_{\tilde{\zeta}})$ and by 2) $\mu(\hat{R}^{\infty}, \mathfrak{A}(\hat{R}^{\infty}, R))=\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, R))$. Thus we have $\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*))\geqq \mu(\hat{R}^{\infty}, \mathfrak{A}(\hat{R}^{\infty}, \underline{R}^*))$. The inverse inequality is clear, because $\hat{\tilde{R}}^{\infty}$ is a covering surface over $\hat{R}^{\infty}$. Therefore

$$\mu(\hat{R}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*))=\mu(\hat{\tilde{R}}^{\infty}, \mathfrak{A}(\hat{\tilde{R}}^{\infty}, \underline{R}^*)).$$

We show that the D-typeness of R does not necessarily imply the D-typeness of $\hat{R}$ by an example.

Example. Let $\{B_{2n}, B_{2n+1}\}$ be domains shown in the figure and construct a holomorphic function of the same kind as in example in "Dirichlet Problem. II". Remove from the unit-circle all the points such that $f(z)=0, 1,$ or 2 and let R be the remaining surface. Then

$$1=\mu(R, \mathfrak{A}(R, \underline{R}^*))>\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*)).$$



If we consider R^∞ as a covering surface $\hat{R}$ over R, we see that $\hat{R}$ is not of D-type, but R is a covering surface of D-type.

From the results obtained till now, we see that the measure $\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*))$ under the condition that the universal covering surface of the projection of R is hyperbolic, depend on the size of $\mathfrak{A}(R, \underline{R}^*)$. The B-typeness and F-typeness depend also on it. Hence theorems 1, 2 and 3 will be natural. On the other hand $\mu(R, \mathfrak{A}(R, \underline{R}^*))$ and D-typeness of R depend not only the size of $\mathfrak{A}(R, \underline{R}^*)$ but on the structure of R and $\mathfrak{A}(R, \underline{R}^*)$, i.e. the class of super-harmonic function $\{v(z)\}$ defining $\mu(R, \mathfrak{A}(R, \underline{R}^*))$. The class is so small that we may have $\mu(R, \mathfrak{A}(R, \underline{R}^*))=1$ on some complicated Riemann surface. Therefore the possibility of the fact that the D-typeness of R does not yield the D-typeness of $\hat{R}$ will be understood.