170. *Uniform Convergence of Fourier Series. III*

By Masako SATÔ
Mathematical Institute, Tokyo Metropolitan University, Japan
(Comm. by Z. SUETUNA, M.J.A., Nov. 12, 1954)

1. **Introduction.** S. Izumi and G. Sunouchi[1] proved the following theorems concerning uniform convergence of Fourier series:

Theorem I. *If*

$$f(t)-f(t')=o\left(1\Big/\log\frac{1}{|t-t'|}\right)\ \text{as}\ t,t'\to x$$

then the Fourier series of $f(t)$ converges uniformly at $t=x$.

Theorem II. *If*

$$f(t)-f(t')=o\left(1\Big/\log\log\frac{1}{|t-t'|}\right)\ \text{as}\ t,t'\to x$$

and the nth Fourier coefficients are $O((\log n)^\alpha/n)$ for $\alpha>0$, then the Fourier series of $f(t)$ converges uniformly at $t=x$.

In this paper, we treat the case that the order of $f(t)-f(t')$ is $o\left(1\Big/\left(\log\frac{1}{|t-t'|}\right)^\alpha\right)$ $(1>\alpha>0)$, $o\left(1\Big/\left(\log\log\frac{1}{|t-t'|}\right)^\alpha\right)$ $(\alpha>0)$ and more generally $o\left(1\Big/\left(\log_k\frac{1}{|t-t'|}\right)^\alpha\right)$.

2. **Theorem 1.** *Let $0<\alpha<1$. If*

$$f(t)-f(t')=o\left(1\Big/\left(\log\frac{1}{|t-t'|}\right)^\alpha\right)\quad(t,t'\to 0)$$

and the nth Fourier coefficients of $f(t)$ is of order $O(e^{(\log n)^\alpha}/n)$, then the Fourier series of $f(t)$ converges uniformly at $t=0$.

Proof. We assume that $x_n\to 0$ and $f(0)=0$.

$$\begin{aligned}S_n(x_n)&=\frac{1}{\pi}\int_0^\pi[f(x_n+t)+f(x_n-t)]\frac{\sin nt}{t}dt+o(1)\\&=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi e^{\beta(\log n)^\alpha}/n}+\int_{\pi e^{\beta(\log n)^\alpha}/n}^{\pi}\right]+o(1)\\&=\frac{1}{\pi}[I+J+K]+o(1),\end{aligned}$$

say, where β is the least number >1 such that $2n\,|\,e^{\beta(\log n)^\alpha}$, then it is sufficient to prove that $s_n(x_n)=o(1)$ as $n\to\infty$.

Since $f(x)$ is continuous, we have $I=o(1)$.

1) S. Izumi and G. Sunouchi: Notes on Fourier analysis (XLVIII): Uniform convergence of Fourier series, Tôhoku Mathematical Journal, **3** (1951).

$$J=\sum_{k=1}^{e^{\beta(\log n)^\alpha}-1}\int_{k\pi/n}^{(k+1)\pi/n}[f(x_n+t)+f(x_n-t)]\frac{\sin nt}{t}\,dt$$

$$=\int_{\pi/n}^{2\pi/n}\sum_{k=0}^{\rho-2}\left[f\left(x_n+t+\frac{k\pi}{n}\right)+f\left(x_n-t-\frac{k\pi}{n}\right)\right]\frac{\sin nt}{t+\frac{k\pi}{n}}dt,$$

where $\rho=e^{\beta(\log n)^\alpha}$. By the first mean value theorem, for $\pi/n\leqq\theta\leqq 2\pi/n$,

$$J=-2\sum_{k=0}^{\rho-2}\frac{(-1)^k}{n\theta+k\pi}\left[f\left(x_n+\frac{k\pi}{n}+\theta\right)+f\left(x_n-\frac{k\pi}{n}-\theta\right)\right]$$

$$=\frac{-2}{\pi}\sum_{k=0}^{(\rho-2)/2}\frac{1}{2k+1}\left[\left\{f\left(x_n+\frac{2k}{n}\pi+\theta\right)-f\left(x_n+\frac{2k+1}{n}\pi+\theta\right)\right\}\right.$$

$$\left.+\left\{f\left(x_n-\frac{2k}{n}\pi-\theta\right)-f\left(x-\frac{2k+1}{n}\pi-\theta\right)\right\}\right]+o(1)$$

$$=o\left(\frac{1}{(\log n)^\alpha}\sum_{k=0}^{(\rho-2)/2}\frac{1}{2k+1}\right)=o\left(\frac{1}{(\log n)^\alpha}\log\rho\right)=o(1).$$

We next prove $K=o(1)$. Now

$$K=2\sum_{\nu=1}^{\infty}a_\nu\cos\nu x_n\int_{\pi e^{\beta(\log n)^\alpha}/n}^{\pi}\cos\nu t\,\frac{\sin nt}{t}\,dt,$$

taking absolute value

$$|K|\leqq 2\sum_{\nu=1}^{\infty}|a_\nu|\left|\int_{\pi e^{\beta(\log n)^\alpha}/n}^{\pi}\frac{\sin(n+\nu)t+\sin(n-\nu)t}{t}\,dt\right.$$

$$=\sum_{\nu=1}^{\infty}|a_\nu|\frac{n}{\pi e^{\beta(\log n)^\alpha}}\left|\int_{\pi e^{\beta(\log n)^\alpha}/n}^{\xi}(\sin(n+\nu)t+\sin(n-\nu)t)dt\right|$$

$$\leqq 2\sum_{\substack{\nu=1\\ \nu\neq n}}^{\infty}|a_\nu|\frac{n}{\pi e^{\beta(\log n)^\alpha}}\cdot\frac{1}{|n-\nu|}+o(1).$$

It is sufficient to prove that

$$\frac{n}{e^{\beta(\log n)^\alpha}}\left[\sum_{\nu=1}^{n-1}\frac{|a_\nu|}{n-\nu}+\sum_{\nu=n+1}^{\infty}\frac{|a_\nu|}{\nu-n}\right]=\frac{n}{e^{\beta(\log n)^\alpha}}[K_1+K_2]=o(1).$$

Now

$$K_1=\sum_{\nu=1}^{[n/2]}\frac{|a_\nu|}{n-\nu}+\sum_{\nu=[n/2]+1}^{n-1}\frac{(a_\nu)}{n-\nu}=O\left(\sum_{\nu=1}^{[n/2]}\frac{e^{(\log\nu)^\alpha}}{\nu(n-\nu)}+\sum_{\nu=[n/2]+1}^{n-1}\frac{e^{(\log\nu)^\alpha}}{\nu(n-\nu)}\right)$$

$$=O\left(\frac{(\log n)^{1-\alpha}}{n}e^{(\log n)^\alpha}+\frac{\log n}{n}e^{(\log n)^\alpha}\right),$$

$$K_2=\left(\sum_{\nu=n+1}^{2n}+\sum_{\nu=2n+1}^{\infty}\right)\frac{|a_\nu|}{\nu-n}=O\left(\frac{e^{(\log n)^\alpha}}{n}\sum_{\nu=n+1}^{2n}\frac{1}{\nu-n}+\sum_{\nu=2n+1}^{\infty}\frac{e^{(\log\nu)^\alpha}}{\nu^2/2}\right)$$

$$=O\left(\frac{\log n}{n}e^{(\log n)^\alpha}+\frac{e^{(\log n)^\alpha}}{n}\right).$$

Accordingly we have

$$K=O\left(\frac{n}{e^{\beta(\log n)^\alpha}}\cdot\frac{\log n\cdot e^{(\log n)^\alpha}}{n}\right)=O\left(\frac{\log n}{e^{\beta(\log n)^\alpha-(\log n)^\alpha}}\right)=o(1).$$

Thus we have $s_n(x_n)=o(1)$ as $n\to\infty$.

3. **Theorem 2.** *Let* $\alpha>1$. *If*

$$f(t)-f(t')=o\left(1\Big/\left(\log\log\frac{1}{|t-t'|}\right)^\alpha\right)\qquad(t,t'\to 0)$$

and the nth Fourier coefficients of $f(t)$ *is of order* $(e^{(\log\log n)^\alpha}/n)$, *then the Fourier series of* $f(t)$ *converges uniformly at* $t=0$.

Proof. As in the proof of Theorem 1, we may assume $x_n\to 0$ and $f(0)=0$.

$$s_n(x_n)=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi e^{(\beta\log\log n)^\alpha}/n}+\int_{\pi e^{\beta(\log\log n)^\alpha}/n}^{\pi}\right]+o(1)$$
$$=\frac{1}{\pi}[I+J+K]+o(1),$$

say, where β is the least number >1 such that $|e^{\beta(\log\log n)^\alpha}$ is odd. Then we have $I=o(1)$ and

$$J=\sum_{k=1}^{e^{\beta(\log\log n)^\alpha}-1}\int_{k\pi/n}^{(k+1)\pi/n}[f(x_n+t)+f(x_n-t)]\frac{\sin nt}{t}dt$$
$$=\int_{\pi/n}^{2\pi/n}\sum_{k=0}^{\xi-2}(-1)^k\left[f\left(x_n+\frac{k\pi}{n}+t\right)+f\left(x_n-\frac{k\pi}{n}-t\right)\right]\frac{\sin nt}{t+\frac{k\pi}{n}}dt,$$

where $\xi=e^{\beta(\log\log n)^\alpha}$. Applying the first mean value theorem,

$$J=-2\sum_{k=0}^{\xi-2}\frac{(-1)^k}{n\theta+k\pi}\left[f\left(x_n+\frac{k\pi}{n}+\theta\right)+f\left(x_n-\frac{k\pi}{n}-\theta\right)\right]$$
$$=-\frac{2}{\pi}\sum_{k=0}^{(\xi-2)/2}\frac{1}{2k+1}\left[\left\{f\left(x_n+\frac{2k}{n}\pi+\theta\right)-f\left(x_n+\frac{2k+1}{n}\pi+\theta\right)\right\}\right.$$
$$\left.+\left\{f\left(x_n-\frac{2k}{n}\pi-\theta\right)-f\left(x_n-\frac{2k+1}{n}\pi-\theta\right)\right\}\right]+o(1)$$
$$=o\left(\frac{1}{(\log\log n)^\alpha}\sum_{k=0}^{(\xi-2)/2}\frac{1}{2k+1}\right)=o\left(\frac{1}{(\log\log n)^\alpha}\log\xi\right)=o(1).$$

We shall next prove that

$$K=2\sum_{\nu=1}^{\infty}a_\nu\cos\nu x_n\int_{\xi\pi/n}^{\pi}\cos\nu t\frac{\sin nt}{t}dt=o(1).$$

Now

$$|K|\leqq\sum_{\nu=1}^{\infty}|a_\nu|\left|\int_{\pi\xi/n}^{\pi}\frac{(\sin(n+\nu)t+\sin(n-\nu)t))}{t}dt\right|$$
$$=\sum_{\nu=1}^{\infty}|a_\nu|\frac{n}{\pi\xi}\left|\int_{\pi\xi/u}^{\eta}(\sin(n+\nu)t+\sin(n-\nu)t)\,dt\right|$$

$$\leqq \frac{2}{\pi}\frac{n}{\xi}\Big[\sum_{\nu=1}^{n-1}\frac{|a_\nu|}{n-\nu}+\sum_{\nu=n+1}^{\infty}\frac{|a_\nu|}{\nu-n}\Big]+o(1)$$

$$=\frac{2}{\pi}\frac{n}{\xi}[K_1+K_2]+o(1)$$

say, then

$$K_1=\Big(\sum_{\nu=1}^{[n/2]}+\sum_{\nu=[n/2]+1}^{n-1}\Big)\frac{|a_\nu|}{n-\nu}=O\Big(\frac{(\log n)^{1-\alpha}}{n}e^{(\log\log n)^\alpha}+\frac{\log n}{n}e^{(\log\log n)^\alpha}\Big),$$

$$K_2=\Big(\sum_{\nu=n+1}^{2n}+\sum_{\nu=2n+1}^{\infty}\Big)\frac{|a_\nu|}{\nu-n}=O\Big(\frac{\log n}{n}e^{(\log\log n)^\alpha}+\frac{e^{(\log\log n)^\alpha}}{n}\Big).$$

Accordingly we get

$$K=O\Big(\frac{n}{e^{\beta(\log\log n)^\alpha}}\cdot\frac{\log n\cdot e^{(\log\log n)^\alpha}}{n}\Big)=o(1).$$

Thus the theorem is proved.

4. **Theorem 3.** *If*

$$f(t)-f(t')=o\Big(1\Big/\psi\Big(\frac{1}{|t-t'|}\Big)\Big)\qquad(t,t'\to 0)$$

and if $f(x)$ is of class $\phi(n)$[2] *then the Fourier series of $f(t)$ uniformly at $t=0$, where $\phi(n)=O(n)$, $\psi(n)=log(n\,\theta(n)/\phi(n))$ and $\theta(n)$ are monotone increasing to infinity as $n\to\infty$.*

Proof. As in the proof of previous theorems we assume $x_n\to 0$ and $f(0)=0$. We put

$$s_n(x_n)=\frac{1}{\pi}\Big[\int_0^{\pi/n}+\int_{\pi/n}^{\beta\theta(n)/\phi(n)}+\int_{\beta\theta(n)/\phi(n)}^{\pi}\Big]+o(1)$$

$$=\frac{1}{\pi}[I+J+K]+o(1),$$

where β is a real number $\geqq 1$ such that $\beta n\theta(n)/\pi\phi(n)$ is an odd integer. Then we have $I=o(1)$, and

$$J=\int_{\pi/n}^{2\pi/n}\sum_{k=0}^{\{\beta n\theta(n)/\pi\phi(n)\}-2}\Big[f\Big(x_n+t+\frac{k\pi}{n}\Big)+f\Big(x_n-t-\frac{k\pi}{n}\Big)\Big]\frac{\sin nt}{t+\frac{k\pi}{n}}dt$$

$$=-2\sum_{k=0}^{\zeta-2}\frac{(-1)^k}{2k+n\theta}\Big[f\Big(x_n+\frac{k\pi}{n}+\theta\Big)+f\Big(x_n-\frac{k\pi}{n}-\theta\Big)\Big]\quad(\pi/n\leqq\theta\leqq 2\pi/n)$$

$$=o\Big(\frac{1}{\psi(n)}\sum_{k=0}^{(\zeta-2)/2}\frac{1}{2k+1}\Big)=o(1),$$

where $\zeta=\beta n\theta(n)/\pi\phi(n)$. We next prove $K=o(1)$. By the second mean value theorem

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2) A function $f(x)$ is said to be of class $\phi(n)$ if

$$\int_a^b f(x+t)\cos nt\,dt=O\,(1/\phi(n))$$

uniformly for all x, n, a, b with $b-a\leqq 2\pi$. *Cf.* J. P. Nash: Rice Institute Pamphlet (1953); M. Satô: Proc. Japan Acad., **30** (1954).

$$K=\frac{\phi(n)}{n\theta(n)}\int_{\beta\theta(n)/\phi(n)}^{\eta}[f(x_n+t)+f(x_n-t)]\sin nt\,dt$$

where $\beta\theta(n)/\phi(n)\leqq\eta\leqq\pi$. Since $\int_a^b f(x+t)\sin nt\,dt=O(1/\phi(n))$, we have

$$K=O\left(\frac{\phi(n)}{\theta(n)}\cdot\frac{1}{\phi(n)}\right)=o(1).$$

This completes the proof of Theorem 3.

Corollary 1. *Let* $0<\alpha<1$. *If*

$$f(t)-f(t')=o\left(1\Big/\left(\log\log\frac{1}{|t-t'|}\right)^{\alpha}\right)\qquad(t,t'\to0)$$

and if $f(x)$ *is of class* $\phi(n)=n/e^{(\log\log n)^{\alpha}}$, *then the Fourier series converges uniformly at* $t=0$.

This follows from Theorem 3, putting

$$\psi\left(\frac{1}{|t-t'|}\right)=\left(\log\log\frac{1}{|t-t'|}\right)^{\alpha},$$

$$\phi(n)=n/e^{(\log\log n)^{\alpha}},\quad \theta(n)=e^{(\beta-1)(\log\log n)^{\alpha}}\ (\beta>1).$$

Corollary 2. *Let* $\alpha>0$ *and* k *be an integer* $\geqq3$. *If*[3]

$$f(t)-f(t')=o\left(1\Big/\left(\log_k\frac{1}{|t-t'|}\right)^{\alpha}\right)$$

and if $f(x)$ *is of class* $\phi(n)=n/e^{(\log n)^{\alpha}}$, *then the Fourier series converges uniformly at* $t=0$.

I wish to express my gratitude to Professor S. Izumi for his suggestions and encouragement.

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3) $\log(\log x)=\log_2 x$, $\log_k(\log x)=\log_{k+1}\times(k\geqq2)$.