203. The Divergence of Interpolations. III

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4. Next we consider a generalization of Theorem 4. Let D be a closed limited points set whose complement K with respect to the extended plane is connected and regular in the sense that K possesses a Green's function with pole at infinity. Let $w=\phi(z)$ map K onto the region $|w|>1$ so that the points at infinity correspond to each other. Let $\Gamma_R(R>1)$ be the level curve determined by $|w|=R>1$.

In this case, we can also define the operation Y_m to $\varphi(z)$ single valued and analytic on Γ_R by

$$(22)\quad \begin{cases} Y_m(\varphi\,;\,a)=\dfrac{\Gamma(1-m)}{2\pi i}\,\mathrm{pf.}\displaystyle\int_{\Gamma_R}\varphi(t)[\phi(t)-\phi(a)]^{m-1}dt; \\ \qquad\qquad\qquad\qquad a \text{ on } \Gamma_R;\ m\neq 1,\ 2,\ldots, \\ Y_m(\varphi\,;\,a)=\dfrac{(-1)^m}{2\pi i}\displaystyle\int_{\Gamma_R}\varphi(t)L_m[\phi(t)-\phi(a)]dt; \\ \qquad\qquad\qquad\qquad a \text{ on } \Gamma_R;\ m=1,\ 2,\ldots, \end{cases}$$

where $(w-\phi(a))^{m-1}$ and $L_m(w-\phi(a))$ are functions single valued and analytic interior to Γ_R which are defined in paragraph 1.

Given a function $f(z)$ which is single valued and analytic throughout the interior of the level curve Γ_R and which has singularities of Y_m type on Γ_R, that is

$$(23)\qquad f(z)=g(z)+\textstyle\sum_{k=1}^{N}g_k(z)y_{m_k}(\phi(z)\,;\,a_k);\quad a_k \text{ on } \Gamma_R,$$

where $g(z)$ and $g_k(z)$ are functions defined by (8) which are single valued and analytic on and within the level curve Γ_R, and $y_{m_k}(w\,;\,a_k)$ are functions defined by (8) which are single valued and analytic interior to Γ_R but have respectively a singularity of Y_m type at $z=a_k$.

Let a set of points (17) lie on D and satisfy the condition that the sequence $W_n(z)/\Delta^n w^n$ converges to an analytic function $\lambda(w)=\lambda(\phi(z))$ non-vanishing for z exterior to D, and uniformly on any closed limited points set exterior to D, and uniformly on any closed limited points set exterior to D, where Δ is *capacity* of D. That is, for any positive number greater than unity,

$$(24)\qquad \lim_{n\to\infty} W_n(z)/\Delta^n w^n=\lambda(w)\neq 0 \text{ uniformly for } |w|\geqq r>1.$$

The sequence of polynomials $S_n(z\,;\,f)$ of respective degrees n found by interpolation to $f(z)$ in all the zeros of $W_{n+1}(z)$ is defined by

$$(25)\qquad S_n(z;f)=S_n(z;g)+\sum_{k=1}^{N}S_n(z;g_k y_{m_k})$$
$$=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{g(t)}{t-z}dt$$
$$+\sum_{k=1}^{N}Y_{m_k}\left(\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{g_k(t)}{t-z};\,a_k\right).$$

By the method similar to the proof of Theorem 3, we have

$$n^{m_k}\left(\frac{\phi(a_k)}{w}\right)^{n+1}S_n(z;g_k y_{m_k})=n^{m_k}\left(\frac{\phi(a_k)}{w}\right)^{n+1}Y_{m_k}\left(\frac{W_{n+1}(t)-W_{n+1}(z)}{W_{n+1}(t)}\frac{g_k(t)}{t-z};\,a_k\right)$$
$$\sim n^{m_k}\phi^{n+1}(a_k)\frac{W_{n+1}(z)}{w^{n+1}}Y_{m_k}\left(\frac{1}{W_{n+1}(t)}\frac{g_k(t)}{t-z};\,a_k\right)$$
$$\sim n^{m_k}\phi^{n+1}(a_k)\lambda(\phi(z))Y_{m_k}\left(w^{-(n+1)}\frac{g_k(t)}{\lambda(\phi(t))(t-z)};\,a_k\right)$$
$$+\,n^{m_k}\phi^{n+1}(a_k)\lambda(\phi(z))Y_{m_k}\left[w^{-(n+1)}\left(\frac{1}{\lambda(w)}-\frac{t^{n+1}}{W_{n+1}(t)}\right)\frac{g_k(t)}{t-z};\,a_k\right]$$
$$\sim(-1)^{m_k}[\phi(a_k)]^{m_k}\lambda(\phi(z))\frac{g_k(a_k)}{\lambda(\phi(a_k))(a_k-z)}=B_k\neq 0,$$

for z exterior to Γ_R.

As a generalization of Theorem 4, a theorem follows by Lemma 3. That is,

Theorem 5. *Let D be a closed limited points set with the capacity Δ whose complement K with respect to the extended plane is connected and regular in the sense that K possesses a Green's function with pole at infinity. Let $w=\phi(z)$ map K onto the region $|w|>1$ so that the points at infinity correspond to each other. Let $W_n(z)$ be the polynomials of respective degrees, n which satisfy the condition (24) and $f(z)$ be a function such that represented by (23).*

Then the sequence of polynomials $S_n(z;f)$ of respective degrees n found by interpolation to $f(z)$ in all the zeros of $W_{n+1}(z)$ diverges at every point exterior to Γ_R. Moreover, we have

$$(26)\qquad \overline{\lim}_{n\to\infty}\left|n^{p}\left(\frac{R}{\phi(z)}\right)^{n}S_n(z;f)\right|>0;\quad |\phi(z)|>R>1,$$

where p is the minimum of real parts of m_k in (23).

Additions and Corrections to Tetsujiro Kakehashi: "The Divergence of Interpolations. I"

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Page 742, equation (5), for "$\frac{1}{2\pi i}$" read "$\frac{(-1)^m}{2\pi i}$".

Page 744, line 4, for "$\lim_{n\to\infty}\frac{1.2\cdots(n-1)}{z(z+1)\cdots(z+n-1)}$"

read "$\lim_{n\to\infty}\frac{1.2\cdots(n-1)}{z(z+1)\cdots(z+n-1)}n^{z}$"