194. Gentzen's Theorem on an Extended Predicate Calculus[1]

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In this paper, I shall show that the corresponding theorem (cf. 2) to Gentzen's 'Hauptsatz'[2] on his 'Kalkül LK' is proved in case of the logical system — I shall call it L_0K — obtained from LK by additional admitting quantifiers $V^0\varphi$ (for all φ) and $\exists^0\varphi$ (there exists φ) where φ is a propositional variable.

1. THE LOGICAL SYSTEM L_0K

1.1. 'Formula'

As the *definition* of formula, we use the *formation rule of formula* obtained from that of the *restricted predicate calculus* — for example, Kalkül LK — by adding to the latter the following item: If $\mathfrak{F}(\alpha)$ is a *formula*, α is a *free propositional variable without argument*, φ is a *bound propositional variable* not contained in $\mathfrak{F}(\alpha)$, and $\mathfrak{F}(\varphi)$ is the result of substituting φ for α throughout $\mathfrak{F}(\alpha)$, then $V^0\varphi\mathfrak{F}(\varphi)$ and $\exists^0\varphi\mathfrak{F}(\varphi)$ are *formulae*.

The *grade* of a formula is the number ($\geqq 0$) of occurrences of logical symbols (&, $\vee$, $\supset$, $\neg$, V, $\exists$, V^0, $\exists^0$) in the formula, and the *degree* the number of occurrences of V^0 and $\exists^0$. For example, a formula of the restricted predicate calculus has the degree 0.

1.2. 'Sequent'

A *sequent* is a formal expression of the form

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$$

where $\mu, \nu \geqq 0$ and $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu, \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ are arbitrary formulae. The part $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu$ is called the *antecedent*, and $\mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ the *succedent* of the sequent.

1.3. 'Rules of inference'

As *rules of inference* we use ones obtained from those for Gentzen's LK, which are represented as the 'Schlussfiguren-Schemata', by adding the following

Additional rules of inference for L_0K

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1) Mr. G. Takeuti had proved otherwise the same result, and afterwards the present proof was obtained.

2) G. Gentzen: Untersuchungen über das logische Schliessen, Math. Zeitschr., **39**, 176-210, 405-431 (1935).

Introduction of	in antecedent	in succedent
V^0:	$\dfrac{\mathfrak{F}(\mathfrak{A}),\ \Gamma \to \Theta}{V^0\varphi\mathfrak{F}(\varphi),\ \Gamma \to \Theta}$	$\dfrac{\Gamma \to \Theta,\ \mathfrak{F}(\alpha)}{\Gamma \to \Theta,\ V^0\varphi\mathfrak{F}(\varphi)}$
$\mathcal{A}^0$:	$\dfrac{\mathfrak{F}(\alpha),\ \Gamma \to \Theta}{\mathcal{A}^0\varphi\mathfrak{F}(\varphi),\ \Gamma \to \Theta}$	$\dfrac{\Gamma \to \Theta,\ \mathfrak{F}(\mathfrak{A})}{\Gamma \to \Theta,\ \mathcal{A}^0\varphi\mathfrak{F}(\varphi)}$

where $V^0\varphi\mathfrak{F}(\varphi)$ or $\mathcal{A}^0\varphi\mathfrak{F}(\varphi)$ is an arbitrary formula of such form, and at this time $\mathfrak{F}(\mathfrak{A})$[3] or $\mathfrak{F}(\alpha)$ is the formula obtained as the result of substituting a formula $\mathfrak{A}$ or a free propositional variable α for the bound propositional variable φ throughout $\mathfrak{F}(\varphi)$, respectively; and Γ and Θ are arbitrary finite sequences of zero or more formulae with separating commas included.

Restriction on variable: The free propositional variable α of the rule should not occur in its conclusion.

The *degree* of V^0-antecedent[4] or $\mathcal{A}^0$-succedent[4] is the degree of the formula $\mathfrak{A}$ of its rule.

1.4. 'Proof'

As the (*formal*) *proof*, we use Gentzen's 'stammbaumförmige Herleitung': it is a finite occurrences of one or more sequents in a partial ordering, which has one lowermost sequent — the *endsequent* — and some uppermost sequents of the form

$$\mathfrak{D} \to \mathfrak{D}$$

where $\mathfrak{D}$ is an arbitrary formula; and in it the premises for inference are written immediately over the conclusion, as in the statement of the rules of inference, and no occurrence of a sequent serves as premise for more than one inference.

The *degree* of a proof is the greatest degree of V^0-antecedent and $\mathcal{A}^0$-succedent contained in the proof if there exists at least one, or is 0 if there is no.

A *proof of* a sequent is a proof which has the sequent as the endsequent. A sequent is said to be *provable* (*with degree* ν), if there exists a proof of the sequent (the degree of which is ν). A sequent is said to be *provable without cut*, if there exists a proof of the sequent which contains no inference conforming the rule

Cut: $$\frac{\Gamma \to \Theta,\ \mathfrak{D} \qquad \mathfrak{D},\ \Delta \to \Lambda}{\Gamma,\ \Delta \to \Theta,\ \Lambda}.$$

3) We consider $\mathfrak{F}(\mathfrak{A})$, only when it is a formula; for example, if $\mathfrak{F}(\varphi)$ is $Vx(F(x)\&\varphi)$, and if $\mathfrak{A}$ is $VxF(x)$, then $\mathfrak{F}(\mathfrak{A})$ is $Vx(F(x)\&VxF(x))$, which is not a formula, where x is a bound individual variable and F is a free predicate variable of one argument place.

4) They are the abbreviations of 'introduction of V^0 in antecedent' and 'introduction of $\mathcal{A}^0$ in succedent', respectively.

2. PRINCIPAL THEOREM

If a sequent is provable, then it is provable also with degree 0 *and without cut* (Gentzen's theorem on L_0K).

3. PROOF OF THE PRINCIPAL THEOREM

For our purpose, it is sufficient to prove two following theorems:

THEOREM 1. If a sequent is provable, then it is provable with degree 0.

THEOREM 2. If a sequent is provable with degree 0, then it is provable with degree 0 and without cut.

3.1. *Proof of Theorem 1*

3.11. LEMMA 1. Each sequent of the following form (3.111.1–3.112.2) is provable with degree 0.

3.111.1. $\mathfrak{A}, \mathfrak{F}(\alpha \vee \neg\alpha) \to \mathfrak{F}(\mathfrak{A})$

3.111.2. $\mathfrak{A}, \mathfrak{F}(\mathfrak{A}) \to \mathfrak{F}(\alpha \vee \neg\alpha)$

3.112.1. $\mathfrak{F}(\mathfrak{A}) \to \mathfrak{F}(\alpha \,\&\, \neg\alpha), \mathfrak{A}$

3.112.2. $\mathfrak{F}(\alpha \,\&\, \neg\alpha) \to \mathfrak{F}(\mathfrak{A}), \mathfrak{A}$

where α is a free propositional variable, and $\mathfrak{F}(\mathfrak{A}), \mathfrak{F}(\alpha \vee \neg\alpha)$ and $\mathfrak{F}(\alpha \,\&\, \neg\alpha)$ are the formulae obtained as the results of substituting formulae $\mathfrak{A}$, $\alpha \vee \neg\alpha$ and $\alpha \,\&\, \neg\alpha$ for a free propositional variable β throughout a formula $\mathfrak{F}(\beta)$, respectively.

Proof is shown by mathematical induction on the grade of $\mathfrak{F}(\beta)$.

3.12. LEMMA 2. Each sequent of the following form (3.121 and 3.122) is provable with degree 0.

3.121. $V^0\varphi\mathfrak{F}(\varphi) \to \mathfrak{F}(\mathfrak{A})$

3.122. $\mathfrak{F}(\mathfrak{A}) \to \exists^0\varphi\mathfrak{F}(\varphi)$

where $V^0\varphi\mathfrak{F}(\varphi)$ or $\exists^0\varphi\mathfrak{F}(\varphi)$ is a formula of such form, and $\mathfrak{F}(\mathfrak{A})$ is the formula obtained as the result of substituting a formula $\mathfrak{A}$ for the bound propositional variable φ throughout $\mathfrak{F}(\varphi)$.

Proof. The (formal) proof of the sequent 3.121 is obtained as follows:

$$\dfrac{\dfrac{\mathfrak{F}(\alpha \,\&\, \neg\alpha) \to \mathfrak{F}(\mathfrak{A}), \mathfrak{A}}{V^0\varphi\mathfrak{F}(\varphi) \to \mathfrak{F}(\mathfrak{A}), \mathfrak{A}}\ V^0\text{-antecedent} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A}, \mathfrak{F}(\alpha \vee \neg\alpha) \to \mathfrak{F}(\mathfrak{A})}{\mathfrak{F}(\alpha \vee \neg\alpha), \mathfrak{A} \to \mathfrak{F}(\mathfrak{A})}\ \text{Interchange}}{V^0\varphi\mathfrak{F}(\varphi), \mathfrak{A} \to \mathfrak{F}(\mathfrak{A})}\ V^0\text{-antecedent}}{\mathfrak{A}, V^0\varphi\mathfrak{F}(\varphi) \to \mathfrak{F}(\mathfrak{A})}\ \text{Interchange}}{\dfrac{\dfrac{V^0\varphi\mathfrak{F}(\varphi), V^0\varphi\mathfrak{F}(\varphi) \to \mathfrak{F}(\mathfrak{A}), \mathfrak{F}(\mathfrak{A})}{V^0\varphi\mathfrak{F}(\varphi) \to \mathfrak{F}(\mathfrak{A}), \mathfrak{F}(\mathfrak{A})}\ \text{Contraction}}{V^0\varphi\mathfrak{F}(\varphi) \to \mathfrak{F}(\mathfrak{A})}\ \text{Contraction}}\ \text{Cut}$$

where over $\mathfrak{A}, \mathfrak{F}(\alpha \vee \neg\alpha) \to \mathfrak{F}(\mathfrak{A})$ and $\mathfrak{F}(\alpha \,\&\, \neg\alpha) \to \mathfrak{F}(\mathfrak{A}), \mathfrak{A}$ we write the proofs of them which have the degree 0 (Lemma 1), respectively. This proof has the degree 0, because two V^0-antecedents explicitly shown in the above figure have both the degree 0.

For 3.122, the treatment is dual to that of 3.121. q.e.d.

3.13. Let us be given a provable sequent, then there exists the proof of the sequent. Choose one of the applications of rules V^0-antecedent and $\mathit{\Xi}^0$-succedent which have the degree >0 —if there exists—, and suppose that it is

$$\frac{\mathfrak{F}(\mathfrak{A}),\ \Gamma \rightarrow \Theta}{V^0\varphi\mathfrak{F}(\varphi),\ \Gamma \rightarrow \Theta}\ V^0\text{-antecedent}$$

(for $\mathit{\Xi}^0$-succedent, treatment is dual). Substitute

$$\frac{V^0\varphi\mathfrak{F}(\varphi) \rightarrow \mathfrak{F}(\mathfrak{A}) \qquad \mathfrak{F}(\mathfrak{A}),\ \Gamma \rightarrow \Theta}{V^0\varphi\mathfrak{F}(\varphi),\ \Gamma \rightarrow \Theta}\ \text{Cut}$$

for that rule, and over $V^0\varphi\mathfrak{F}(\varphi) \rightarrow \mathfrak{F}(\mathfrak{A})$ write the proof of it which has the degree 0 (Lemma 2). Repeat the procedure until all of the rules which have the degree >0 have been changed, then the proof of the given sequent which has the degree 0 is obtained.

3.2. *Proof of Theorem 2* is obtained without difficulty from Gentzen's proof of his Hauptsatz on *LK* by using mathematical induction on the degree of the formula, within both the basis and the induction step of which mathematical induction is used on the grade of the formula, instead of that on the grade of the formula in the latter.