7. *Harmonic Measures and Capacity of Sets of the Ideal Boundary. II*

By Zenjiro KURAMOCHI

Mathematical Institute, Osaka University

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Let R be a positive boundary Riemann surface and let D[1] be a non compact domain determining a subset B_D of the ideal boundary. Put $D_n=(R-R_n)\cap D$. Let $U_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-D_n$ such that $U_{n,n+i}(z)=0$, on ∂R_0, $U_{n,n+i}(z)=1$ on ∂D_n and $\frac{\partial U_{n,n+i}}{\partial n}=0$ on $\partial R_{n+i}-D_n$. Then $\lim\limits_{n=\infty}\lim\limits_{i=\infty} U_{n,n+i}(z)=\lim\limits_{n=\infty} U_n(z)=U(z)$, where $U(z)$ is the equilibrium potential of B_D. We have proved that

$$\int\limits_{\partial R_0}\frac{\partial U_n}{\partial n}ds=\int\limits_{\partial G_\varepsilon}\frac{\partial U_n}{\partial n}ds \tag{1}$$

for every G_ε except for at most one ε, where G_ε is the domain in which $U_n(z)>1-\varepsilon$. Let $U'_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-G_\varepsilon-R_0$ such that $U'_{n,n+i}(z)=0$ on ∂R_0, $U'_{n,n+i}(z)=1-\varepsilon$ on $\partial G_\varepsilon\cap R_{n+i}$ and $\frac{\partial U'_{n,n+i}}{\partial n}=0$ on $\partial R_{n+i}-G_\varepsilon$. Then $\lim\limits_{i=\infty} U'_{n,n+i}(z)=U_n(z)$. Since every $U'_{n,n+i}(z)=1-\varepsilon$ on ∂G_ε, $\frac{\partial U'_{n,n+i}}{\partial n}\to\frac{\partial U_n}{\partial n}$: $\frac{\partial U'_{n,n+i}}{\partial n}\leqq 0$ on every point of $\partial G_\varepsilon\cap R_{n+i}$. Hence by (1) and $\lim\limits_{i=\infty}\int\limits_{\partial R_0}\frac{\partial U_{n,n+i}}{\partial n}ds=\int\limits_{\partial R_0}\frac{\partial U_n}{\partial n}ds$, we easily that

$$\lim_{i=\infty}\int\limits_{\partial G_\varepsilon}\varphi_i\frac{\partial U'_{n,n+i}}{\partial n}ds=\int\limits_{\partial G_\varepsilon}\varphi\frac{\partial U_n}{\partial n}ds \tag{2}$$

on ∂G_ε for every bounded sequence of continuous functions $\varphi_i\to\varphi$: $|\varphi_i|\leqq M<\infty$.

We denote by G_n the domain in which $U_n(z)>1-\varepsilon_n$, where $\varepsilon_1>\varepsilon_2>\cdots$; $\lim\limits_n \varepsilon_n=0$ and every ε_n satisfies the condition (1).

Let $U''_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-G_n$ such that $U''_{n,n+i}(z)=U(z)$ on $\partial G_\varepsilon+\partial R_0$ and $\frac{\partial U''_{n,n+i}}{\partial n}=0$ on $\partial R_{n+i}-G_n$. Since $U_n(z)$ is the function such that $U_n(z)=1-\varepsilon_n$ and $U_n(z)$ has the minimum Dirichlet integral over $R-R_0-G_n$, and since $\lim\limits_{n=\infty} U_n(z)=U(z)$ on ∂G_n, then by (2) we can prove as in the previous paper[2]

$$\lim_{n=\infty}\lim_{i=\infty} U''_{n,n+i}(z)=U(z).$$

1) See, the definition of non compact domain. "Harmonic measures and capacity. I".

2) See (1).

Hence we have the following

Lemma.

$$U(z)=U_{ex}(z),$$

where the extremisation is with respect to the sequence $\{G_n\}$.

Now we apply Green's formula to $U'_{n,n+i}(z)$ and $U''_{n,n+i}(z)$. Then

$$\int\limits_{\partial R_0+(\partial G_n\cap R_{n+i})+(\partial R_{n+i}-G_n)} U'_{n,n+i}(z)\frac{\partial U''_{n,n+i}}{\partial n}ds=\int\limits_{\partial R_0+(\partial G_n\cap R_{n+i})+(\partial R_{n+i}-G_n)} U''_{n,n+i}(z)\frac{\partial U'_{n,n+i}}{\partial n}ds \quad \text{and}$$

$$(1-\varepsilon_n)\int\limits_{\partial R_0}\frac{\partial U''_{n,n+i}}{\partial n}ds=\int\limits_{\partial G_n\cap R_{n+i}} U''_{n,n+i}(z)\frac{\partial U'_{n,n+i}}{\partial n}ds.$$

Let $i\to\infty$. Then by (1) and (2) we have

$$(1-\varepsilon_n)\int\limits_{\partial R_0}\frac{\partial U}{\partial n}ds=\int\limits_{\partial G_n} U(z)\frac{\partial U_n}{\partial n}ds. \tag{3}$$

On the other hand, since $\lim\limits_{n=\infty}\int\limits_{\partial R_0}\frac{\partial U_n}{\partial n}ds=\mathrm{Cap}\,(B_D)=\int\limits_{\partial R_0}\frac{\partial U}{\partial n}ds$ and $(1-\varepsilon_n)\int\limits_{\partial R_0}\frac{\partial U_n}{\partial n}ds=\int\limits_{\partial G_n}U_n(z)\frac{\partial U_n}{\partial n}ds$, we have by (3)

$$\lim_{n=\infty}\int\limits_{\partial G_n}(U_n(z)-U(z))\frac{\partial U_n}{\partial n}ds=\lim_{n=\infty}(1-\varepsilon_n)\int\limits_{\partial G_0}\Big(\frac{\partial U_n}{\partial n}-\frac{\partial U}{\partial n}\Big)ds=0. \tag{4}$$

Since $\lim\limits_{i=\infty}\underset{R_{n+i}-R_0-G_n}{D}(U'_{n,n+i}(z))=\underset{R-R_0-G_n}{D}(U_n(z))$ and $\lim\limits_{i=\infty}\underset{R_{n+i}-R_0-G_n}{D}(U''_{n,n+i}(z))=\underset{R-R_0-G_n}{D}(U(z))$, we have

$$\underset{R-R_0-G_n}{D}(U_n(z)-U(z),\ U_n(z))=\lim_{i=\infty}\underset{R_{n+i}-R_0-G_n}{D}(U'_{n,n+i}(z)-U''_{n,n+i}(z),\ U_{n,n+i}(z))$$
$$=\int\limits_{\partial G_n}(U_n(z)-U(z))\frac{\partial U_n}{\partial n}ds.$$

Hence by (4) $\quad \lim\limits_{n=\infty}\underset{R-R_0-G_n}{D}(U_n(z)-U(z),\ U_n(z))=0.\quad$ Thus

$$\underset{R-R_0-G_n}{D}(U_n(z)-U(z))=\underset{R-R_0-G_n}{D}(U(z))-D(U_n(z)),\quad \text{whence}$$

$$\lim_{i=\infty}\underset{R-R_0-G_n}{D}(U(z))\geqq\lim_{\substack{n=\infty\\ R-R_0-G_n}}D\,(U_n(z)). \tag{5}$$

Since $\underset{G_n-D_n}{D}(U_n(z))=\underset{R-R_0-D_n}{D}(U_n(z))-\underset{R-R_0-G_n}{D}(U_n(z))=\varepsilon_n\int\limits_{\partial R_0}\frac{\partial U_n}{\partial n}ds$,

$$\lim_{n=\infty}\underset{G_n-D_n}{D}(U_n(z))=0. \tag{6}$$

From the Fatou's Lemma, we have

$$\underset{R-R_0}{D}(U(z))=\underset{R-R_0}{D}(\lim_{n=\infty}U_n(z))\leqq\lim_{n=\infty}(\underset{R-R_0-D_n}{D}(U_n(z))=\mathrm{Cap}\ (B_D).$$

Therefore by (5) and (6), we have $\lim\limits_{n=\infty}\underset{G_n-D_n}{D}(U(z))=0$. Hence $\lim\limits_{n=\infty}\underset{R-G_n-R_0}{D}(U_n(z))=\lim\limits_{n=\infty}\underset{R-G_n-R_0}{D}(U(z))$, $\lim\limits_{n=\infty}\underset{G_n-D_n}{D}(U_n(z))=\lim\limits_{n=\infty}\underset{G_n-D_n}{D}(U(z))=0$ and

$$\lim_{n=\infty}\underset{G_n-D_n}{D}(U_n(z)-U(z),\ U_n(z))=0.$$

Therefore

$$\lim_{n=\infty} \underset{R-R_0}{D}(U_n(z)-U(z)) = \lim_{n=\infty} (\underset{R-G_n}{D}(U_n(z)-U(z)) + \underset{G_n-D_n}{D}(U_n(z)-U(z)) = 0.$$

It follows that $U_n(z)$ converges to $U(z)$ in norm. Then we have the following

Proposition. Cap $(B_D) = \int_{\partial R_0} \frac{\partial U}{\partial n} ds = \underset{R-R_0}{D}(U(z))$.

The extremisation is defined with respect to the sequence $\{G_n\}$, we can also the above operation with respect to $\{D_n\}$.

Every $U_m(z)\,(m=n,\ n+1,\dots)$ $(U(z)=\lim U_n(z))$ is the harmonic function which has the minimum Dirichlet integral over $R-R_0-D_n$ among all functions which have their boundary value $U_m(z)$ on ∂D_n. Let $h(z)$ be a harmonic function in $R-R_0-D_n$ such that $h(z)=0$ on $\partial D_n+\partial R_0$ and $\underset{R-D_n}{D}(h(z)) \leqq M < \infty$. Then

$$D(U_m(z) \pm \varepsilon h(z)) \leqq D(U_m(z)) \pm 2\varepsilon\ D(U_m(z),\ h(z)) + \varepsilon^2 D(h(z)),$$

whence

$$\underset{R-R_0-D_n}{D}(h(z),\ U_m(z)) = 0.$$

Let $\tilde{U}_n(z)$ be a harmonic function in $R-R_0-D_n$ such that $\tilde{U}_n(z)=U(z)$ on $\partial D_n+\partial R_0$ and $\hat{U}_n(z)$ has the minimum Dirichlet integral over $R-R_0-D_n$.

Then $\underset{R-D_n-R_0}{D}(\tilde{U}_n(z)) \leqq \underset{R-R_0}{D}(U(z))$ and $\underset{R-R_0-D_n}{D}(\tilde{U}_n(z), h(z)) = 0.$

Since $\lim_{n=\infty} U_n(z) = U(z)$ on ∂D_n and $\lim_{n=\infty} \underset{R-R_0-D_n}{D}(U_n(z)-U(z)) = 0$,

we can assume $h(z) = \tilde{U}_n(z) - U(z)$. Then we have

$$\lim_{m=\infty} [\underset{R-R_0-D_n}{D}(U(z)-U_m(z),\ h(z))]^2 \leqq \lim_{m=\infty} [\underset{R-R_0-D_n}{D}(h(z))\underset{R-R_0-D_n}{D}(U(z)-U_m(z))] = 0.$$

Hence $\underset{R-R_0-D_n}{D}(U(z), h(z)) = 0$, therefore

$$0 = \underset{R-R_0-D_n}{D}(U(z)-\tilde{U}_n(z), h(z)) = \underset{R-R_0-D_n}{D}(U(z)-\tilde{U}_n(z)),\ \text{whence}\ U(z) = \tilde{U}_n(z).$$

Thus we have the next

Theorem 4.

$$U(z) = U_{ex}(z),$$

where the extremisation is defined with respect to the sequence $\{D_n\}$.

Corollary 1. If $U(z) \not\equiv 0$, $\overline{\lim_{z \in D}}\ U(z) = 1$.

Proof. Let $\hat{U}_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-D_n$ such that $\hat{U}_{n,n+i}(z) = U(z)$ on $\partial D_n \cap R_{n+i}$, $\hat{U}_{n,n+i}(z) = 0$ on ∂R_0 and $\frac{\partial \hat{U}_{n,n+i}}{\partial n} = 0$ on $\partial R_{n+i} - D_n$. Then $\tilde{U}_n(z) = \lim_{i=\infty} \hat{U}_{n,n+i}(z)$. Assume $U(z) \leqq K < 1$ on D. Then $\hat{U}_{n,n+i}(z) \leqq K\, U_{n,n+i}(z)$. Hence

$$U(z) = \lim_{n=\infty}\lim_{i=\infty} \hat{U}_{n,n+i}(z) \leqq K \lim_{n=\infty}\lim_{i=\infty} U_{n,n+i}(z) = K\, U(z).$$

This is absurd. Hence $\overline{\lim\limits_{z\in D}}\, U(z)=1$.

Corollary 2. If $U(z)\not\equiv 0$, then $\overline{\lim}\, U(z)=1$ in B_D except possibly for a subset of B_D of outer capacity zero.

We denote by $J_\lambda(\lambda<1)$ the domain where $U(z)<\lambda$. Put $D\cap J_\lambda=H_\lambda$. Then H_λ is a non compact domain determining a subset I_λ of B_D. Let $U_I(z)$ be the equilibrium potential of I_λ. Then it is clear that $U_I(z)\leqq U(z)$. Hence $\overline{\lim\limits_{z\in H_\lambda}}\, U_I(z)\leqq\lambda$. Therefore by the above corollary $U_I(z)\equiv 0$.

On the Behaviour of the Green's Function in the Neighbourhood of the Ideal Boundary

Let $G(z, z_0)$ be the Green's function of R and let M be sufficiently large number. Then $G_M=\mathop{\mathcal{E}}\limits_{z}\{G(z, z)>M\}$ is compact. We can suppose $R_0=G_M$. If we consider $R-R_0$ as a non compact domain D defining all ideal boundary of R. Then it is clear that

$$1-\frac{G(z, z_0)}{M}=U(z)=\omega'(z),$$

where $U(z)$ and $\omega'(z)$ is the equilibrium potential and harmonic measure. Then by the corollary $U(z)=1$ except possibly a subset of ideal boundary of capacity zero. Let $D_\lambda=\mathop{\mathcal{E}}\limits_{z}\{U(z)>\lambda\}$ be a non compact domain determining B_D. Let $U_\lambda(z)$, $\omega'_\lambda(z)$ and $\omega_\lambda(z)$ be equilibrium potential of B_D and harmonic measures. Then $0=U_\lambda(z)=\omega'_\lambda(z)$ and $\omega_\lambda(z)=0$ is equivalent to $\omega(z)=0$. Thus we have the next

Theorem 5. Cap $(B_D)=0=\omega_\lambda(z)$.

We can construct an open Riemann surface $\hat{D}_\lambda$ by the process of symmetrization with respect to ∂D_λ. Then we have the following

Corollary. *$D_\lambda+\hat{D}_\lambda$ is a null-boundary Riemann surface.*

Proof. Let $\omega_n(z)$ be the harmonic measure of $(\partial R_n\cap D_\lambda)+(\partial R_n\cap D_\lambda)^{\wedge}$ with respect to $((D_\lambda\cap R_n)-R_0)+((D_\lambda\cap R_n)-R_0)^{\wedge}$. Then $\omega_n(z)=0$ on $\partial R_0+\partial \hat{R}_0$, $\omega_n(z)=1$ on $(\partial R_n\cap D_\lambda)$ and $\frac{\partial\omega_n}{\partial n}=0$ on ∂D_λ. On the other hand let $U_{n,n+i}(z)$ be a function in $(D_\lambda\cap R_n)-R_0$ such that $U_{n,n+i}(z)=0$ on ∂R_0, $U_{n,n+i}(z)=1$ on $\partial D_\lambda\cap(R-R_n)$ and $\frac{\partial U_{n,n+i}}{\partial n}=0$ on $\partial R_{n+i}-D_\lambda$. Then it is clear that $\mathop{D}\limits_{(D_\lambda\cap R_n)-R_0}(\omega_n(z))\leqq \mathop{D}\limits_{R_{n+i}-R_0}(U_{n,n+i}(z))$. Hence, since B_D is a set of capacity zero, we have

$$\mathop{D}\limits_{D_\lambda\cap R-R_0}(\lim\omega_n(z))\leqq\mathop{D}\limits_{R-R_0}(\lim_{n=\infty}\lim_{i=\infty}U_{n,n+i}(z))=0.$$

Thus $D_\lambda+\hat{D}_\lambda$ is a null-boundary Riemann surface.

Corollary. *Let $G(z, z_0)$ be the Green's function of R and let $h(z)$*

be its conjugate. Put $W(z)=e^{-G(z,z_0)-ih(z)}=re^{i\theta}$. *We cut* R *along the trajectories* ($h(z)=const$) *so that* $W(z)$ *may be single valued. Then* R *is mapped onto the domain* $|W(z)|<1$ *with enumerably infinite number of radial slits. Then* $z=z^{-1}(W)$ *can be continued analytically along radii* $re^{i\theta}$ *from* $W=0$ *to* $|W|=1$ *except possibly a set of* θ *of angular measure zero.*

In fact, if it were not so, there exists a set I_λ of the ideal boundary such that I_λ is defined by a non compact domain $D_\lambda=\underset{z}{\mathcal{E}}\{G(z, z)>\lambda\}$ and the length of the image of C enclosing I_λ is larger than $l\,(>0)$. Since Cap $(I_\lambda)=0$, there exists a harmonic function $U_n(z)$ in $R-(R_n\cap D_\lambda)-R_0$ such that $\int_{C\mu}\frac{\partial U_n}{\partial n}ds=2\pi$ and $U_n(z)=M_n$ ($\lim_{n=\infty}M_n=\infty$) on $\partial R_n\cap D$, where $C\mu=\underset{z}{\mathcal{E}}\{U_n(z)=\mu\}$. Thus by usual method we can deduce a contradiction. Analogously we have

Corollary. *If the analytic function* $f(z)$ *satisfies* $D_R(f(z))<\infty$. *Then the length of the image of trajectories mapped by* $f(z)$ *is finite for almost* θ.

Applications to the Subregion on an Abstract Riemann Surface

Let D be a non compact domain in R. If any bounded (Dirichlet Bounded) harmonic function vanishing on ∂D or having vanishing normal derivative on ∂D must reduce to a constant, we denote by S_{0B}, S_{0D}, S_{0NB} and S_{0ND} such class of D respectively. In the previous paper,[3] we have proved that, if D can be mapped onto a bounded domain then, $S_{0NB}\subset S_{0B}$.

Theorem 6. *If the genus of* D *is finite, then* $D\in S_{0NB}=S_{0ND}$ *is equivalent to that* $D+\hat{D}$ *is a null-boundary Riemann surface.*

Proof. If $D+\hat{D}$ is a null-boundary Riemann surface, it is clear that $D\in S_{0NB}\,(S_{0ND})$. By assumption, we can suppose $D-R_{n_0}$ is a planer surface. Assume $D+\hat{D}$ is a positive boundary Riemann surface. Then the harmonic measure $\omega(z)$ of the ideal boundary of $(D-R_{n_0})+(D-R_{n_0})\hat{}$ is non-constant. Normalize $\omega(z)$ so that $\int_{\partial R_0}\frac{\partial\omega'(z)}{\partial n}=2\pi$ and let $h(z)$ be its conjugate. Then $e^{\omega'(z)+ih(z)}=W(z)$ maps $D-R_{n_0}$ onto the domain $1<|W|<K$ with enumerably infinite number of radial slits which are the images of ∂D such that ∂R_{n_0} is mapped onto $|W|=1$ and $(D-R_{n_0})\hat{}$ is symmetric to $(D-R_{n_0})$ with respect to these slits.

Let D_λ ($\lambda<K$) be the domain in which $W(z)<\lambda$. Then D_λ determines a set of ideal boundary of capacity zero. Thus we can easily

3) Z. Kuramochi: On covering surfaces, Osaka Math. Jour. (1953).

prove that $\int_{\partial D_\lambda}\frac{\partial\omega'}{\partial n}ds=\int_{\partial R_0}\frac{\partial\omega'}{\partial n}ds$, whence the length of the image of $\partial D_\lambda=2\cdot 2\pi\lambda$. Let G be a non compact domain of $(D-R_{n_0})+(D-R_{n_0}\hat{)}$ lying over $1<|W|<K$ and $0<\arg W<\pi$. Let $U_n(z)$ be a harmonic function in $(D-R_{n_0}-(G\cap D_\lambda))\ (D-R_{n_0}-(G\cap D_\lambda)\hat{)}$ such that $U_n(z)=0$ on $\partial R_{n_0}+\partial\hat{R}_{n_0}$, $U_n(z)=1$ on $\partial D_\lambda\cap G$ and has the minimum integral. On the other hand let $U'_n(z)$ be a harmonic function in the ring $1\leqq|W|<\lambda$ with radial slits above-mentioned such that $U'_n(W)=1$ on $|W|=\lambda$, $0<\arg W<\pi$ and $U'_n(W)$ has the minimum Dirichlet integral. Then

$$\underset{D-R_{n_0}}{D}(U_n(z)) \geqq 2\underset{1<|W|<\lambda}{D}(U'_n(W)) \geqq \frac{2\pi}{\log\lambda}.^{4)}$$

Therefore by theorem

$$D(\lim U_n(z))=\mathrm{Cap}\,(B_D)\geqq\frac{2\pi}{\log K}>0.\quad \text{Hence } \lim_n \underset{z\in G}{\overline{U}_n(z)}=1.$$

On the other hand, let $V_n(W)$ be a harmonic function on the ring without radial slits such that $V_n(W)=1$ on $|W|=\lambda$, $0<\arg W<\pi$ and $V_n(W)$ has the minimum Dirichlet integral. Then clearly

$$D(U(z)) \leqq 2D(\lim_n V_n(W))<2D(\omega(z))=\frac{4\pi}{\log K}.$$

Therefore on $D+\hat{D}$, there exists a non-constant bounded and Dirichlet bounded harmonic function, because, if it were not so $U(z)$ must be a multiple of $\omega(z)$.

4) Z. Kuramochi: On the behaviour of analytic functions on abstract Riemann surfaces to appear in Ann. Sci. Acad. Fenn.