5. Vector-space Valued Functions on Semi-groups. I

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S. Bochner and J. v. Neumann (1) extended the theory of almost periodic functions to functions whose values are elements of vector spaces. In recent papers, W. Maak (2), (3) discussed almost periodic functions defined on semi-groups. W. Maak theory may be regarded a generalisation of J. v. Neumann theory of almost periodic functions on groups (4). In these Notes, we shall consider vector valued functions on semi-groups.

By a semi-group G, we shall mean an algebraic system in which a multiplication is defined, and the law of composition has the associative property. A locally convex space E is a topological vector space over real field in which there is a fundamental system of neighborhoods (briefly n.b.d.) of O which are convex. A locally convex space E is (F)-space, if it is metrisable and complete. Throughout what follows, all functions considered are to be mappings of a semi-group G into a locally convex space E.[*]

I. The definition of almost periodic functions

Definition 1. A function $f(x)$ is called almost periodic, if, for a given n.b.d. U,[1] *there are finite family of subsets $A_1, A_2, \dots, A_n$ of G such that*

(1) $$G=\bigcup_{i=1}^{n} A_i$$

(2) $$c'x'd',\ c'y'd' \in A_i \text{ implies } f(cxd)-f(cyd) \in U \text{ for all } c, d \in G.$$

By the decomposition $\{f(x), U\}$ of G, we shall mean the sets $\{A_i\}$ $(i=1, 2, \dots, n)$ in Definition 1. It is clear that any constant function on G is almost periodic.

Theorem 1. Let G be a group and $f(x)$ vector valued function. The function $f(x)$ is almost periodic, if and only if, for any n.b.d. U, there are subsets $A_1, A_2, \dots, A_n$ of G such that

(1)′ $$G=\bigcup_{i=1}^{n} A_i$$

(2)′ $x, y \in A_i$ *implies* $f(cxd)-f(cyd) \in U$ *for* c, d *of* G.

Proof. If $f(x)$ is almost periodic, it is clear that a decomposition $\{f(x), U\}$ satisfies the conditions (1)′ and (2)′.

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*) For details of locally convex spaces, see J. Dieudonné: Recent developments in the theory of locally convex vector spaces, Bull. Amer. Math. Soc., **59**, 495–512 (1953).

1) n.b.d. U means convex n.b.d. U.

Conversely, suppose that there are subsets $A_i (i=1, 2, \dots, n)$ satisfying (1)$'$, (2)$'$. For some c_0, d_0, x, y of G and c_0xd_0, c_0yd_0 of G, we have

$$f(cc_0xd_0d)-f(cc_0yd_0d) \in U$$

for all c, d. Since G is a group, for c, d of G, we have

$$f(cxd)-f(cyd) \in U.$$

Therefore $\{f(x), U\}$ is a decomposition of G.

Theorem 2. Let G be a group, and f(x) vector valued function on G. f(x) is almost periodic, if for any n.b.d. U, there are subsets A_i $(i=1, 2, \dots, n)$ *of G such that*

(1)$''$ $$G=\bigcup_{i=1}^{n} A_i$$

(2)$''$ *for some* x, y *of* G *and* $x, y \in A_i$, *we have*

$$f(cx)-f(cy) \in U \quad \text{for all} \quad c \in G.$$

Proof. For any n.b.d. U, we can find subsets A_i of G satisfying (1)$''$, (2)$''$. Then we define $B_{i_1 i_2}, \dots, {}_{i_n}$ as the set $\bigcap_{k=1}^{n} A_{i_k} a_k^{-1}$, where $a_k \in A_k$ and $i_k=1, 2, \dots, n$. Let $x, y \in B_{i_1}, \dots, {}_{i_n}$ and $d \in A_k$, then xa_k, $ya_k \in A_{i_k}$. Hence by

$$f(cxd)-f(cyd)=(f(cxd)-f(cxa_k))+(f(cxa_k)-f(cya_k)) + (f(cya_k)-f(cyd)),$$

we have

$$f(cxd)-f(cyd) \in 3U.$$

This shows that $f(x)$ is almost periodic.

Theorem 3. The set R of all almost periodic functions on G forms a closed linear space for the uniform topology.

Proof. It is obvious that R is a linear space. To prove that R is closed, let $f_n(x) \Rightarrow f(x)$ i.e., for any n.b.d. U, there is an index n_0 such that

$$f_n(x)-f(x) \in U$$

for $n \geq n_0$ and $x \in G$. Let $A_i (i=1, 2, \dots, n)$ be a decomposition for G by f_{n_0} and U, then for $c'xd', c'yd' \in A_i$, we have

$$f(cxd)-f(cyd)=(f(cxd)-f_{n_0}(cxd))+(f_{n_0}(cxd)-f_{n_0}(cyd)) + (f_{n_0}(cyd)-f(cyd)) \in 3U.$$

Thus $f(x)$ is almost periodic.

II. Ergodic function and its mean

Definition 2. f(x) is said to be left-ergodic, if, for any n.b.d. U, there are an element f of E and elements $a_1, a_2, \dots, a_n$ *of G such that*

(3) $$f-\frac{1}{n}\sum_{i=1}^{n} f(a_i d) \in U$$

for all d of G. Similarly we can define right-ergodic, $f(x)$ is said *to be right-ergodic,* if for any n.b.d. U, there are an element g of

E and elements $b_1, b_2, \dots, b_n$ of G such that

$$(4) \qquad g - \frac{1}{n}\sum_{i=1}^{n} f(cb_i) \in V$$

for all c of G.

If $f(x)$ is left-ergodic and right-ergodic, it is said to be ergodic. We shall call the f in (3) U-left mean of $f(x)$, and g in (4) V-right mean of $f(x)$. Then we have

Theorem 4. If $f(x)$ is ergodic, and f, g are U-left mean, V-right mean of $f(x)$ respectively, then

$$f - g \in U + V.$$

Especially, if E is a Banach space with norm $\|f\|$, then we have

Theorem 5. If $f(x)$ is ergodic and f_ε, $g_{\varepsilon'}$ are ε-left, ε'-right means of $f(x)$ respectively, then

$$\|f_\varepsilon - g_{\varepsilon'}\| < \varepsilon + \varepsilon'.$$

Therefore, if $f_{\frac{1}{n}}$ is $\frac{1}{n}$-left mean,

$$\|f_{\frac{1}{n}} - g_{\varepsilon'}\| < \frac{1}{n} + \varepsilon'.$$

Hence

$$\|f_{\frac{1}{m}} - f_{\frac{1}{n}}\| < \frac{1}{n} + \frac{1}{m} + 2\varepsilon'.$$

This shows that $\left\{f_{\frac{1}{n}}\right\}$ is Cauchy sequence. Let f be the limit of $\left\{f_{\frac{1}{n}}\right\}$, then $\|f - g_{\varepsilon'}\| \le \varepsilon'$. Similarly, $g_{\frac{1}{n}}$ $(n=1, 2, \dots)$ is convergent, if g is the limit, we have $f = g$. Thus we have the following

Theorem 6. Let $f(x)$ be a Banach space valued function. If it is ergodic, then there is an element f such that for any positive number ε,

$$\|f - \frac{1}{n}\sum_{i=1}^{n} f(a_i d)\| < \varepsilon$$

and

$$\|f - \frac{1}{m}\sum_{j=1}^{m} f(cb_j)\| < \varepsilon$$

for some a_i $(i=1, 2, \dots, n)$, b_j $(j=1, 2, \dots, m)$.

Such an f is uniquely determined and we shall call f the mean value of $f(x)$, and denote it by $M(f)$.

Using the semi-norm on a locally convex complete space E, we have the following

Theorem 7. Let $f(x)$ be an E-valued function. If it is ergodic then there is an element $M(f)$ of E such that, for any n.b.d. U,

$$M(f) - \frac{1}{n}\sum f(a_i d) \in U$$

and

$$M(f)-\frac{1}{m}\sum f(cb_j)\in U$$

for some $a_i\ (i=1, 2,\dots, n),\quad b_j\ (j=1, 2,\dots, m).$

Such $M(f)$ is called the *mean of* $f(x)$. Thus

Corollary. Every ergodic function has one and only one mean.

References

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4) J. v. Neumann: Almost periodic functions on groups, Trans. Amer. Math. Soc., **36**, 445–492 (1934).