34. Note on an Extension of Multiplication of Distributions

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(Comm. by K. KUNUGI, M.J.A., March 12, 1955)

Multiplication of distributions are considered by L. Schwartz (1) in his text-book in case only when one of the distributions is a non-function at each step of multiplication. Indeed, according to the ordinary definition we can not consider other sort of multiplication.

Meanwhile in some applied branches (for example in the calculations of S-matrix by the power series expansions of coupling constant (J. Schwinger (2), F. J. Dyson (3), etc.)), we meet usually rather extraordinary multiple several numbers of whose multiplicand are non-function distributions. So it will be desired to examine the possibility of extension of its definition to the case in which more than two non-functions can occur. (We shall need such examinations also in the case when we want to examine whether or not we are able to consider the non-linear equation whose solution is a non-function distribution.)

Recently L. Schwartz (4) has pointed out the impossibility of the associative multiplication including δ and the derivative operation from purely algebraic consideration. In this paper we study the extended multiplication mainly from the topological consideration. That is to say, if the multiplying operation $T \rightarrow QT$ by a fixed distribution Q, is defined by the contragradient mappings $\varphi \rightarrow \varphi Q$ for $\varphi \in \mathfrak{D}$, then the structure of the space $\mathfrak{Q} = \{\varphi Q \mid \varphi \in \mathfrak{D}\}$ determines the nature of the multiplication. So if we require some conditions for the topology of the space $\mathfrak{Q}$ (whose algebraic structure is assumed to be the same as in the space $\mathfrak{D}'$), we can determine the range of the multiplicands and the multiples independently from the other algebraic requirements such as the law of the derivation or of the association except the linearity of the multiplication which is always assured by this sort of definition.

The main result of this paper studied along this line is the following: Considering two multiplicands, if we impose a condition (C) upon the extended multiplication, then we can consider at most a multiple T such that either T is essentially an ordinary multiple of Q and of $\alpha \in \mathfrak{D}$ or T is a limit in $\mathfrak{D}'$ of ordinary multiple $\alpha_\lambda Q$.

Concerning the terminologies used in this paper, see for example L. Schwartz (1), N. Bourbaki (5), and J. Dieudonné (6).

1. For a fixed distribution $Q(\in \mathfrak{D}')$ we consider the vector space

$\mathfrak{Q}=\{\varphi Q \mid \varphi \in \mathfrak{D}\}$ which is the subspace of $\mathfrak{D}'$, where φQ means the ordinary multiplication in $\mathfrak{D}'$.

We consider further the condition:

(A) the locally convex topological vector space $\mathfrak{P}$ contains both $\mathfrak{D}[=\{\alpha\varphi \mid \varphi \in \mathfrak{D},\ \alpha \in \mathfrak{E}\}]$ and $\mathfrak{Q}$.

Now we define the extended multiplication $\circ$ by

(B) $\langle T, \varphi\alpha\rangle_{\mathfrak{P}',\mathfrak{P}} = \langle \alpha \circ T, \varphi\rangle$ for $\varphi \in \mathfrak{D}$, $\alpha \in \mathfrak{E}$,

$\langle T, \varphi Q\rangle_{\mathfrak{Q}',\mathfrak{Q}} = \langle Q \circ T, \varphi\rangle$ for $\varphi \in \mathfrak{D}$,

where $\mathfrak{P}'$ is the dual space of $\mathfrak{P}$ and $\langle\, ,\rangle_{\mathfrak{P}',\mathfrak{P}}$ is the scalar product between these spaces.

In order to construct the space $\mathfrak{P}$ so that the right hand side of (B) means $\langle\, ,\rangle_{\mathfrak{D}',\mathfrak{D}}$ (i.e. $Q \circ T$ or $\alpha \circ T$ defines a distribution), the following condition (C)$'$ is obviously necessary and sufficient.

(C)$'$ $\langle T, \varphi Q\rangle$ or $\langle T, \varphi\alpha\rangle$ is a continuous linear functionals for $\varphi \in \mathfrak{D}$.

In our case, the linearity is automatically satisfied if we construct the spaces $\mathfrak{P}$ and $\mathfrak{P}'$ as above. Concerning the continuity we take a rather strong sufficient condition (C).

(C) The mappings from $\mathfrak{D}$ to $\mathfrak{P}$ (C_1) $\varphi \to \varphi\alpha$, (for any fixed $\alpha \in \mathfrak{E}$) and (C_2) $\varphi \to \varphi Q$ are continuous.

The bilinearity of the above constructed multiplication is assured in the following sense

$$(\alpha +_{\mathfrak{P}} Q)\circ T = (\alpha\circ T) +_{\mathfrak{D}'} (Q\circ T)$$
$$\alpha\circ(T_1 +_{\mathfrak{P}'} T_2) = (\alpha\circ T_1) +_{\mathfrak{D}'} (\alpha\circ T_2)$$
$$Q\circ(T_1 +_{\mathfrak{P}'} T_2) = (Q\circ T_1) +_{\mathfrak{D}'} (Q\circ T_2).$$

Here $+_{\mathfrak{A}}$ means the additive operation in the vector space $\mathfrak{A}$.

Further we require the condition our multiplication $\circ$ being an extension of the ordinary one.

(D) $\mathfrak{E}$ is imbedded in $\mathfrak{P}'$ preserving its ordinary multiplication i.e. for any $\beta \in \mathfrak{E}$ and $\alpha \in \mathfrak{E}$, $Q\circ\beta = \beta Q$ and $\alpha\circ\beta = \beta\alpha$.

Now the set of topological spaces $\mathfrak{A}_\alpha$ with topologies τ_α makes a semi-ordered set in the meaning such that $(\tau_\alpha, \mathfrak{A}_\alpha) > (\tau_\beta, \mathfrak{A}_\beta)$ means $\mathfrak{A}_\beta$ contains $\mathfrak{A}_\alpha$ and τ_α is finer than the relative topology $\tau_{\beta,\alpha}$ of τ_β on $\mathfrak{A}_\alpha$. In this case the dual space $\mathfrak{A}'_\alpha$ does not always contain $\mathfrak{A}'_\beta$ but if we consider the restriction $\mathfrak{A}'_{\beta,\alpha}$ defined on $\mathfrak{A}_\alpha$ of $\mathfrak{A}'_\beta$ then we see $\mathfrak{A}'_\alpha \supset \mathfrak{A}'_{\beta,\alpha}$. So, in order that we may find the maximal range of the multiplicands by our scheme it is necessary and sufficient to require the following condition (E).

(E) $\mathfrak{P}$ is the least space which satisfies the conditions (A), (C), and (D) having the finest locally convex topology.

Temporarily in place of (E) we require

(E_1) $\mathfrak{P}$ is the least space which satisfies the condition (A).

(E_2) $\mathfrak{P}$ has the finest (if comparable) locally convex topology which satisfies the conditions (C_1) and (C_2).

2. From the conditions (A) and (E_1), we see immediately that $\mathfrak{P}$ is the sum of the spaces $\mathfrak{D}$ and $\mathfrak{Q}$. Denoting $\mathfrak{D}\frown\mathfrak{Q}$ by $\mathfrak{Q}_0$ and the vector space $\{(q, -q)\mid q\in\mathfrak{Q}_0\}$ by $\mathfrak{N}_0$ we see the vector space $\mathfrak{P}(=\mathfrak{D}+\mathfrak{Q})$ is algebraically isomorphic to the quotient space $\mathfrak{N}=(\mathfrak{D}\oplus\mathfrak{Q})/\mathfrak{N}_0$ where $\oplus$ means the direct sum.

Now if we denote the direct sum topology $\tau_{\mathfrak{P},\mathfrak{D}\oplus\mathfrak{P},\mathfrak{Q}}$ of the two topologies $\tau_{\mathfrak{P},\mathfrak{D}}$ and $\tau_{\mathfrak{P},\mathfrak{Q}}$ in the direct sum space $\mathfrak{Q}\oplus\mathfrak{D}$ and its quotient topology (by $\mathfrak{N}_0$) by $\tau_{\mathfrak{P},\mathfrak{D}+\mathfrak{P},\mathfrak{Q}}$ then we can see that the topology $\tau_{\mathfrak{P},\mathfrak{D}+\mathfrak{P},\mathfrak{Q}}$ of the space $\mathfrak{N}$ is finer than the topology $\tau_{\mathfrak{N}}$ of the space $\mathfrak{N}$ which is homeomorphic to $\tau_{\mathfrak{P}}$.

For any neighborhood $V_{\mathfrak{N}}$ of $\mathfrak{N}_0$ there exist $U_{\mathfrak{P}}$ and $W_{\mathfrak{P}}$ such that for $V_{\mathfrak{P}}=f^{-1}(V_{\mathfrak{N}})$, where f means the above isomorphic mapping from $\mathfrak{P}$ on $\mathfrak{N}$,

$$V_{\mathfrak{P}}\supset U_{\mathfrak{P}}+W_{\mathfrak{P}}\supset U_{\mathfrak{P}}\frown\mathfrak{D}+W_{\mathfrak{P}}\frown\mathfrak{Q}.$$

Since $f(d+q)=(f_1(d), f_2(q))+\mathfrak{N}_0$ for $d\in\mathfrak{D}$, $q\in\mathfrak{Q}$, we have

$$V_{\mathfrak{N}}\supset(f_1(U_{\mathfrak{P}}\frown\mathfrak{D}), f_2(W_{\mathfrak{P}}\frown\mathfrak{Q}))+\mathfrak{N}_0.$$

Here $f_1(d)=Pr_{\mathfrak{D}}f(d)$, $f_2(q)=Pr_{\mathfrak{Q}}f(q)$, for $d\in\mathfrak{D}$, $q\in\mathfrak{Q}$, where $Pr_{\mathfrak{D}}$ or $Pr_{\mathfrak{Q}}$ means the projection to the space $\mathfrak{D}$ or $\mathfrak{Q}$, and $(\mathfrak{A}, \mathfrak{B})$ means the subsets $\{(a, b)\mid a\in\mathfrak{A}, b\in\mathfrak{B}\}$ in the space $\mathfrak{D}\oplus\mathfrak{Q}$.

We can also see that if $\tau_{\mathfrak{D}}$ and $\tau_{\mathfrak{Q}}$ are finer than $\tau_{\mathfrak{P},\mathfrak{D}}$ and $\tau_{\mathfrak{P},\mathfrak{Q}}$ respectively, then the above sum topology $\tau_{\mathfrak{D}+\mathfrak{Q}}$ is finer than the topology $\tau_{\mathfrak{P},\mathfrak{D}+\mathfrak{P},\mathfrak{Q}}$. Moreover if $\tau_{\mathfrak{D}}$ is the locally convex topology of the vector space $\mathfrak{D}$ which satisfies the condition (C_1), and if $\tau_{\mathfrak{Q}}$ is the locally convex topology of the vector space $\mathfrak{Q}$ which satisfies the condition (C_2), then the space $\mathfrak{P}$ which is the inverse image $f^{-1}(\mathfrak{N})$ of the space $\mathfrak{N}$ having sum topology $\tau_{\mathfrak{D}+\mathfrak{Q}}$ satisfies the conditions (A), (C_1), and (C_2), since $\tau_{\mathfrak{P},\mathfrak{D}}\leq\tau_{\mathfrak{D}}$ and $\tau_{\mathfrak{P},\mathfrak{Q}}\leq\tau_{\mathfrak{Q}}$. Thus we obtain the following lemma.

Lemma 1. *If there exist a finest locally convex topology $\tau_{\mathfrak{D}}$ and $\tau_{\mathfrak{Q}}$ which satisfy (C_1), (C_2) then the space $\mathfrak{P}$ which corresponds to the space $\mathfrak{N}$ having the sum topology $\tau_{\mathfrak{D}+\mathfrak{Q}}$ satisfies the conditions (A), (C_1), (C_2), (E_1), and (E_2).*

Now the topology of the space $\mathfrak{D}$ in Lemma 1 is nothing but the ordinary topology of the space $\mathfrak{D}$ by the conditions (C_1) and (E_2), so we have only to find the finest locally convex topology of the space $\mathfrak{Q}$ in order to construct the desired topological vector space $\mathfrak{P}$.

We denote the polar of the space $\mathfrak{Q}$ (in the space $\mathfrak{D}$) by $\mathfrak{Q}^0$ i.e. $\mathfrak{Q}^0=\{\varphi\,\big|\,|\langle\varphi, \psi Q\rangle|\leq 1$ for any $\psi\in\mathfrak{D}\}$ (which is the same $\mathfrak{Q}^0=\{\varphi\mid\langle\varphi, \psi Q\rangle=0$ for any $\psi\in\mathfrak{D}\}$), then $\mathfrak{Q}^0$ is a weakly closed convex subspace (therefore closed convex subspace) of the topological vector space [(5)]. The algebraic structure of $\mathfrak{Q}$ is isomorphic to the factor space $\mathfrak{D}/\mathfrak{Q}^0$. So the finest locally convex topology of $\mathfrak{Q}$ which satisfies the condition (C_2) is the one which is homeomorphic to the ordinary quotient topology $\tau_{\mathfrak{D}/\mathfrak{Q}^0}$.

Now we can cancel the phrase "if comparable" in the conditions (E_2), and we have the following lemma.

Lemma 2. *The finest locally convex topological vector space which satisfies the conditions (A), (C_1), (C_2) is the above defined $\mathfrak{P}$:*

$$\mathfrak{P}=f^{-1}(\mathfrak{N}),\quad \tau_{\mathfrak{N}}=\tau_{\mathfrak{D}+\mathfrak{Q}}=\tau_{(\mathfrak{D}\oplus\mathfrak{Q})/\mathfrak{N}_0}$$

where the space $\mathfrak{Q}$ has the topology such that $\mathfrak{Q}$ is isomorphic to the topological vector space $\mathfrak{D}/\mathfrak{Q}^0$ (having quotient topology) and the space $\mathfrak{D}$ has the ordinary topology.

3. Since $\mathfrak{N}=(\mathfrak{D}\oplus\mathfrak{Q})/\mathfrak{N}_0$, $\mathfrak{N}'$ is isomorphic to the orthogonal (=polar) space $(\mathfrak{N}_0)^0$ of the space $\mathfrak{N}_0$ in the direct product space $\mathfrak{D}'\oplus\mathfrak{Q}'$. The scalar product for an element $n=(\varphi_1, \varphi_2Q)+\mathfrak{N}_0$ of $\mathfrak{N}$ and an element $n'=(n_1', n_2')$ of $\mathfrak{N}'$, is given by

$$\langle n', n\rangle_{\mathfrak{N}',\mathfrak{N}}=\langle n_1', \varphi_1\rangle_{\mathfrak{D}',\mathfrak{D}}+\langle n_2', \varphi_2Q\rangle_{\mathfrak{Q}',\mathfrak{Q}},$$

where $\varphi_1, \varphi_2\in\mathfrak{D}$ and $n_1'\in\mathfrak{D}'$, $n_2'\in\mathfrak{Q}'$.

The dual space $\mathfrak{Q}'$ is isomorphic to the space $(\mathfrak{D}/\mathfrak{Q}^0)'$, and the isomorphic mappings h_1, h_2 between these spaces are given as follows:

h_1 maps $\overline{\varphi}(\in\mathfrak{D}/\mathfrak{Q}^0)$ to $\varphi Q(\in\mathfrak{Q})$ and h_2 maps $T(\in(\mathfrak{D}/\mathfrak{Q}^0)')$ to $h_2(T)$ $(\in\mathfrak{Q}')$ defined by the following equation (j).

(j) $$\langle h_2(T), \varphi Q\rangle_{\mathfrak{Q}',\mathfrak{Q}}=\langle h_2(T), h_1(\overline{\varphi})\rangle_{\mathfrak{Q}',\mathfrak{Q}}$$
$$=\langle T, \overline{\varphi}\rangle_{(\mathfrak{D}/\mathfrak{Q}_0)', (\mathfrak{D}/\mathfrak{Q}_0)}=\langle T, \varphi\rangle_{\mathfrak{D}',\mathfrak{D}}\,.$$

Now for $\beta(x)Q$ taken as T where $\beta(x)\in\mathfrak{E}$, we see $\langle h_2(\beta Q), \varphi Q\rangle_{\mathfrak{Q}',\mathfrak{Q}}=\langle\beta Q, \varphi\rangle_{\mathfrak{D}',\mathfrak{D}}=\langle\beta, \varphi Q\rangle_{\mathfrak{E},\mathfrak{E}'}$, for any $\varphi Q\in\mathfrak{Q}$. To examine the condition (D) in the space $\mathfrak{N}'$ we identify $(\beta, h_2(\beta Q))\in\mathfrak{N}'$ with $\beta\in\mathfrak{E}$. Then we can see (D) is satisfied as follows:

$$\langle Q\circ\beta, \varphi\rangle_{\mathfrak{D}',\mathfrak{D}}=\langle\beta, \varphi Q\rangle_{\mathfrak{N}',\mathfrak{N}}$$
$$=\langle\beta, 0\rangle_{\mathfrak{D}',\mathfrak{D}}+\langle h_2(\beta Q), \varphi Q\rangle_{\mathfrak{Q}',\mathfrak{Q}}=\langle\beta Q, \varphi\rangle_{\mathfrak{D}',\mathfrak{D}} \text{ by (j)},$$

and $$\langle\alpha\circ\beta, \varphi\rangle_{\mathfrak{D}',\mathfrak{D}}=\langle\beta, \alpha\varphi\rangle_{\mathfrak{N}',\mathfrak{N}}$$
$$=\langle\beta, \alpha\varphi\rangle_{\mathfrak{D}',\mathfrak{D}}+\langle h_2(\beta Q), 0\rangle_{\mathfrak{Q}',\mathfrak{Q}}=\langle\beta\alpha, \varphi\rangle_{\mathfrak{D}',\mathfrak{D}}.$$

So we see that the space $\mathfrak{P}'$ satisfies (A), (B), (C), (D), and (E).

As we have seen hitherto, the range $\mathfrak{P}'$ of the possible multiplicands is determined by the requirements (A), (B), (C), and (E) while the condition (D) is satisfied by the space $\mathfrak{P}'$. Similarly the following properties hold in the space $\mathfrak{P}'$ without special requirements.

(1) $T(\in\mathfrak{P}')$ restricted on $\mathfrak{D}$ defines a distribution n_1' such that $T=g(n_1', n_2')$, and the equality $\alpha\circ g(n_1', n_2')=\alpha n_1'$ holds on $\mathfrak{D}$ for any $\alpha\in\mathfrak{E}$ where g denotes the isomorphic mapping from $\mathfrak{N}'$ to $\mathfrak{P}'$.

(2) Similarly to the identifying of $\beta\in\mathfrak{E}$ we can identify some $T\in\mathfrak{D}'$ with $(T, h_2(TQ))\in\mathfrak{N}'$ in case when TQ is a well defined dis-

tribution in the meanings of the ordinary multiplication (for example when $T \in \mathfrak{E}^{(m)}$ and $Q \in \mathfrak{D}^{(m)\prime}$, or $T \in \mathfrak{D}^{(m)\prime}$ and $Q \in \mathfrak{E}^{(m)}$ etc.). In this case of course it holds $\alpha \circ g(T, h_2(TQ)) = \alpha T$ for any multiplicable $\alpha \in \mathfrak{D}'$.

(3) The mapping from $(\mathfrak{E} \times \mathfrak{P}')$ to $\mathfrak{D}'$ $(\alpha, T) \to \alpha \circ T$ is separately continuous.

(4) The mapping from $\mathfrak{P}'$ to $\mathfrak{D}'$, $T \to Q \circ T$ is continuous. This is seen from the fact that $\{\varphi Q \mid \varphi \in B_{\mathfrak{D}}\}$ is a bounded set in $\mathfrak{Q}$ where $B_{\mathfrak{D}}$ denotes a bounded set in the space $\mathfrak{D}$.

(5) Considering in the space $\mathfrak{P}'_{Q_1} \cap \mathfrak{P}'_{Q_2}$ obviously $(Q_1 + Q_2) \circ T = Q_1 \circ T + Q_2 \circ T$, where P_{Q_i} denotes the space $\mathfrak{P}$ which corresponds to Q_i.

(6) The Leibnitz's formula holds under the following definition of derivative in $\mathfrak{P}'$ in the case when each argument has the meaning. For $T \in \mathfrak{P}'$, we define DT by $\langle DT, p \rangle_{\mathfrak{P}', \mathfrak{P}} = \langle T, D^* p \rangle_{\mathfrak{P}', \mathfrak{P}}$ for any $p \in \mathfrak{P}$ where $D^* p$ means the conjugate derivative of p in the meaning of the one in $\mathfrak{D}'$.

(7) We have hitherto discussed the extended multiplication $Q_1 \circ Q_2$ with two multiplicands. But we can iterate this multiplication taking the distribution $Q_1 \circ Q_2$ as Q again and generally we can consider a multiplication with n multiplicands $((\cdots (Q_1 \circ Q_2) \circ Q_3) \circ \cdots \circ Q_n)$.

We see that this iterated extended multiplication can occur in the case when one of the distributions is a non-function distributions at each step of multiplication, using the identification mentioned in (2) and we see its value coincides with the ordinary one. Therefore we can see that the associative law does not always hold. For example $(\delta \circ x) \circ 1/x = 0$ and $\delta \circ (x \circ 1/x) = 1$, where roughly speaking, $(\delta \circ x) \circ 1/x$ means $\{\delta \circ (x, x)\} \circ (1/x, 1/x)$, and $\delta \circ (x \circ 1/x)$ means $\delta \circ (\{x \circ (1/x, 1/x)\}, \{x \circ (1/x, 1/x)\})$.

Remark. h_2 corresponds to the division operator by Q, and the expression $h_2(\beta Q)$ or $h_2(TQ)$ is not necessarily uniquely determined for βQ or TQ though $\langle h_2(\beta Q), \varphi Q \rangle$ or $\langle h_2(TQ), \varphi Q \rangle$ has a definite value. For example $h_\delta(x\delta) = \phi(x)$ for any $\phi : \phi(0) = 0$, and $h_x(\psi(x)x) = \psi(x) + c\delta$ for any constant c, where h_Q denotes the mapping h_2 concerning Q.

4. Any element $p'(= g(n_1', n_2'))$ of the space $\mathfrak{P}'$ is multiplicable by $\alpha + Q (\alpha \in \mathfrak{E})$ and its value is

$$\langle p', \alpha + \varphi Q \rangle_{\mathfrak{P}', \mathfrak{P}} = \langle n_1', \alpha \rangle_{\mathfrak{D}', \mathfrak{D}} + \langle n_2', \varphi Q \rangle_{\mathfrak{Q}', \mathfrak{Q}}.$$

As we see here, so far as the possibility of the multiplication concerns we have only to consider the spaces $\mathfrak{Q}$ and $\mathfrak{Q}'$, though we were dealing until now with the spaces $\mathfrak{P}$ and $\mathfrak{P}'$. In fact, we can discuss about the least space $\mathfrak{Q}$ with the finest topology under the sole requirement (C_1) in a similar way as above.

Now the space $\mathfrak{Q}'$ is isomorphic to the space $\bar{\mathfrak{Q}}$ since $(\mathfrak{D}/\mathfrak{Q}^0)'$ is isomorphic to the orthogonal subspace $\mathfrak{Q}^{00}$ of $\mathfrak{Q}^{0}$ in the space $\mathfrak{D}'$ and $\mathfrak{Q}^{00}$ coincides with the closure $\bar{\mathfrak{Q}}$ of the space $\mathfrak{Q}$ [(5)].

Any element S_2 of the space $\mathfrak{Q}'$ (or $S=g(S_1, S_2)$ of $\mathfrak{P}'$) can be multiplied by Q and defines the multiple $S_2\circ Q$ (or $S\circ Q$) which belongs to $\bar{\mathfrak{Q}}\subset\mathfrak{D}'$. Conversely any element T of the space $\bar{\mathfrak{Q}}$ is considered a multiple of Q by certain element $h_2(T)(\in\mathfrak{P}')$ as already seen in (j). Especially if T belongs to the subspace $\mathfrak{Q}$ of $\bar{\mathfrak{Q}}$, putting $T=\beta Q$ where $\beta\in\mathfrak{E}$, we see $h_2(T)$ can be identified with β as above and this coincides with the ordinary multiplication. Thus we have the following

Theorem. *A multiple $T\circ Q$ of $T(\in\mathfrak{Q}'$ (or $\in\mathfrak{P}'$)) by $Q(\in\mathfrak{D}')$ which satisfies the condition (C_1) (or (A), (C_1), and (C_2)) is possible if and only if $QT\in\bar{\mathfrak{Q}}$. In other words we can consider a multiple $Q\circ T$ if and only if $Q\circ T$ is an ordinary multiple αQ of Q by $\alpha(\in\mathfrak{E})$ or $Q\circ T$ is a (strong) limit in $\mathfrak{D}'$ of ordinary multiple $\alpha_\lambda Q$.*

Under the same requirements as in the theorem we have following two corollaries.

Corollary 1. *If $\mathfrak{Q}$ is closed in $\mathfrak{D}'$ then essentially wider extended multiplication $\circ$ is impossible.*

Corollary 2. *$T\in\mathfrak{D}'$ is multiplicable ($\circ$) by Q if and only if $T\in h_Q(\bar{\mathfrak{Q}})$.*

Remark 1. Similar consideration is possible also in the case $\mathfrak{D}^{(m)}$ and $\mathfrak{D}^{(m)\prime}$.

Remark 2. The theorem combined with other theorems regarding $\mathfrak{Q}^{00}$ will give other corollaries or criterions. For example, combined with the theorem by L. Schwartz (7) "Pour que la distribution T d'ordre m (fini ou infini) soit solution de la famille d'équations multiplicative $Q_j^0 T=0$ (où Q_j^0 sont une famille de fonctions de ($\mathfrak{E}$) qui satisfont l'équation multiplicative $Q_j^0\, Q=0$), il faut et il suffit qu'elle soit limite dans ($\mathfrak{D}'$) ou faiblement dans $\mathfrak{D}^{(m)\prime}$ de combinaisons linéaires finies de distributions à supports ponctuels d'ordre$\leq m$, solutions de la famille d'équations", our theorem gives a criterion of T being a multiple $S\circ Q$ of Q.

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