32. Note on the Isomorphism Problem for Free Algebraic Systems

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Let $V=\{\alpha_1, \alpha_2, \dots\}$ be the system of single-valued compositions, and A the family of composition-identities with respect to V, and let $E=\{a_1, a_2, \dots\}$ be the free generator system. Then it is easily verified that the free A-algebraic system $A^V(E)$, or shortly $A(E)$, can be defined. Let $F=\{b_1, b_2, \dots\}$ be another free generator system. If the cardinal numbers of E and F are equal, then it is clear that $A(E)\cong A(F)$.*)

In this note we shall show that, under some conditions of $A(E)$, $A(E)\cong A(F)$ if and only if the cardinal numbers of E and F are equal, i.e. we shall give the solution of the isomorphism problem for the free A-algebraic system satisfying such conditions. And the isomorphism problems of free groups, free lattices, and others can be easily solved as the special cases of our results.

Theorem I. *Let $A(E)$ be a free A-algebraic system satisfying the following two conditions:*

1) *the composition-identity $x=y$ is not derived from A,*

2) *the cardinal number of E is infinite.*

Then $A(E)\cong A(F)$ if and only if the cardinal numbers of E and F are equal.

Proof. "If"-part of this theorem is immediate. Hence we shall prove "only if"-part.

Let $E=\{a_1, a_2, \dots\}$ and $F=\{b_1, b_2, \dots\}$. Now suppose that $\overline{E}>\overline{F}$**) in spite of $A(E)\cong A(F)$. First we can suppose $A(E)=A(F)$ instead of $A(E)\cong A(F)$, without loss of generality. Hence $b_1, b_2, \dots$ are represented by finite compositions of finite elements in E respectively, i.e.

$$b_1=f_1(E),\ b_2=f_2(E), \dots .$$

Let E' be the set of all the elements in E which appear in some $f_i(E)$. Then the cardinal number of E' is smaller than $\overline{E}$. Hence there exists an element $a_j\in E$ such that $a_j\notin E'$. And a_j is also represented by finite compositions of finite elements in F, i.e. $a_j=\varphi(F)$. Putting $f_1(E), f_2(E), \dots$ in places of $b_1, b_2, \dots$ respectively, we get $a_j=\psi(E)$. Taking off unnecessary elements from E in this identity, we get $a_j=\psi(E'')$, where E'' is a finite set contained in E'.

*) K. Shoda: Allgemeine Algebra, Osaka Math. J., **1** (1949).

**) $\overline{E}$ denotes the cardinal number of E.

Now the relation $a_j=\psi(E'')$ is derived from A, since any relation among the elements in E is derived from A. Since a_j is independent of E'', we get $c=\psi(E'')$ for any element c in $A(E)$. Hence all elements in $A(E)$ are equal, i.e. the composition-identity $x=y$ is derived from A. This is a contradiction.

In the case of $\overline{E}<\overline{F}$, a contradiction is similarly obtained as above. Hence $\overline{E}=\overline{F}$. This completes our proof.

Remark 1. In the case that the composition-identity $x=y$ is derived from A, $A(E)$ consists of only one element. Hence in this case the isomorphism problem is trivial.

Remark 2. In the case that all the compositions are unary, i.e. mappings only, if we assume only the condition 1), then we can get the same result as in Theorem I, in the same way as its proof.

Theorem II. *Let $A(E)$ be the free A-algebraic system. If there exists A^* containing A such that*

3) *the composition-identity $x=y$ is not derived from A^*,*

4) *if E is a finite set, then $A^*(E)$ is also finite,*

then $A(E)\cong A(F)$ if and only if the cardinal numbers of E and F are equal.

Proof. "If"-part of this theorem is immediate. Hence we shall prove "only if"-part.

Since the composition-identity $x=y$ is not derived from A^*, $x=y$ is not derived from A, i.e. A satisfies the condition 1) in Theorem I. Hence in the case that E is infinite, this theorem is immediate.

In the following we shall prove this theorem in the case that E is finite. Let $E=\{a_1,\dots,a_m\}$, $F=\{b_1,\dots,b_n\}$ and $A(E)\cong A(F)$. Then, of course, $A^*(E)\cong A^*(F)$, and n is finite by Theorem I. Now suppose that $m>n$ in spite of $A^*(E)\cong A^*(F)$. Then

$$A^*(a_1,\dots,a_n)\cong A^*(b_1,\dots,b_n)\cong A^*(a_1,\dots,a_n,a_{n+1},\dots,a_m).$$

Hence we get $a_{n+1},\dots,a_m\in A^*(a_1,\dots,a_n)$, using the finiteness of the elements of $A^*(a_1,\dots,a_n)$ and $A^*(a_1,\dots,a_n,a_{n+1},\dots,a_m)$. Accordingly a_m is represented by finite compositions of some elements in $\{a_1,\dots,a_n\}$, i.e. $a_m=f(a_1,\dots,a_n)$. The relation $a_m=f(a_1,\dots,a_n)$ is derived from A^*, since any relation among the elements $a_1,\dots,a_n,a_m$ is derived from A^*. Hence we get $c=f(a_1,\dots,a_n)$ for any element c in $A^*(E)$. Accordingly the composition-identity $x=y$ is derived from A^*. This is a contradiction.

In the case of $m<n$, a contradiction is similarly obtained as above. Hence $m=n$. This completes our proof.

Remark 3. The assumption in Theorem II is satisfied by free lattices, free loops, free semi-groups, free groups, free rings, and others.