
By Masakiti KINUKAWA
Mathematical Institute, Tokyo Metropolitan University, Tokyo
(Comm. by Z. SUETUNA, M.J.A., March 12, 1955)

1. Let $f(x)$ be an integrable function with period $2\pi$ and its Fourier series be

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} A_n(x).$$

We call the series

$$B_n(t) = \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)$$

$$A'_n(t) = \sum_{n=1}^{\infty} n(a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} nA_n(x)$$

conjugate series, derived series and conjugate derived series of (1.1), respectively.

The infinite series $\sum a_n$ is said to be summable by Riesz’s logarithmic mean of order $\alpha$, or simply summable $(R, \log, \alpha)$, to sum $s$, provided that

$$R_\alpha(\omega) = \frac{1}{(\log \omega)^\alpha} \sum_{n=1}^{\infty} (\log \omega/n)^\alpha a_n$$

tends to a limit $s$, as $\omega \to \infty$.

The summability by Riesz’s logarithmic means of the Fourier series was treated by Hardy [1], Takahashi [3], and Wang [4], [5], [6]. Wang has proved the Riesz summability analogue of Bosanquet’s theorem concerning Cesàro summability of Fourier series. This theorem was extended to the derived Fourier series by Matsuyama [2]. In this paper we shall prove the analogue for the conjugate derived Fourier series and some related theorems.

We shall introduce some notations. Let us put

$$g_0(t) = g(t),$$

$$g_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} \left(\frac{\log \frac{u}{t}}{u} \right)^{\alpha-1} \frac{g(u)}{u} \, du \quad (\alpha > 0).$$

Then $g_\alpha(t) / \left(\log \frac{1}{t}\right)^\alpha$ is called the Riesz logarithmic mean of $g(t)$ of order $\alpha$. If the Riesz logarithmic mean of $g(t)$ tends to zero as $t \to 0$, then we write

$$\lim_{t \to 0} g(t) = s \ (R, \log, \alpha).$$
We denote by \( g_\beta(t) \) the \( \beta \)th integral of \( g(t) \), that is,
\[
g_\beta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - u)^{\beta-1} g(u) du,
\]
\( g_\beta(t) = g_\beta(t) \).

2. In what follows we put
\[
\varphi(t) = \varphi(x, t) = \frac{1}{2} \left\{ f(x + t) + f(x - t) - 2t \cos t \right\},
\]
\[
g(t) = \int_t^\infty \frac{\varphi(u)}{u^s} du
\]
and suppose that \( \varphi(t)/t \) is integrable in \((0, \infty)\). Then our theorems are stated as follows.

**Theorem 1.** If
\[
limit_{t \to 0} g(t) = 0 \quad (R, \log, \alpha),
\]
then the conjugate derived Fourier series of \( f(t) \) is summable \((R, \log, \alpha + 2)\) to \( s \) at the point \( x \), where \( \alpha \geq 0 \).

**Theorem 2.** If we suppose
\[
\int_0^t g_\beta(u) du = g_\beta(t) = o \left( t \left( \log \frac{1}{t} \right)^s \right)
\]
and
\[
\int_t^\infty \frac{g_\beta(u - t) - g_\beta(u)}{u} du = o \left( \left( \log \frac{1}{t} \right)^{s+2} \right),
\]
then the conjugate derived Fourier series of \( f(t) \) is summable \((R, \log, \alpha + 2)\) to \( s \) at the point \( x \), where \( \alpha \geq 0 \).

**Theorem 3.** If the conjugate derived Fourier series is summable \((R, \log, \alpha)\), then we have
\[
limit_{t \to 0} g(t) = 0 \quad (R, \log, \alpha + 1 + \varepsilon),
\]
where \( \alpha \geq 2 \) and \( \varepsilon \) is a positive number.

3. We start by some lemmas which need for the proof of our theorems.  \(^1\)

**Lemma 1.** Let us put
\[
S_\alpha(t) = \int_0^1 \left( \log \frac{1}{u} \right)^\alpha \sin \alpha u du
\]
for \( \alpha > -1 \). Then we have the following relations:
\[
\begin{align*}
S_\alpha(t) &= \left\{ \begin{array}{ll}
O(1) & \text{for } t > 0 \text{ and } \alpha > -1, \\
O[\left( \log t \right)^\alpha/t] & \text{for } t \geq 2, \alpha \geq 0, \\
O[\left( \log t \right)^{\alpha-1}/t] & \text{for } t \geq 2, 0 < \alpha < 1,
\end{array} \right.
\end{align*}
\]
\[
S'_\alpha(t) = \left\{ \begin{array}{ll}
O[\left( \log t \right)^\alpha/t^2] & \text{for } t \geq 2, \alpha \geq 1, \\
O(1/t^{1+\alpha}) & \text{for } t \geq 2, 0 \leq \alpha < 1,
\end{array} \right.
\]
\[
S''_\alpha(t) = O[\left( \log t \right)^\alpha/t^3] & \text{for } t \geq 2, \alpha \geq 1,
\]
\[
S_\alpha(0) = 0 & \quad (\alpha > -1),
\]
\(^1\) Cf. Matsuyama [2] and Wang [4], [6], [7].
(3.5) \[ S_0(t) = \frac{1}{2} (1 - \cos t) t, \]
(3.6) \[ [tS_0(t)]' = \alpha S_{-1}(t) \quad \text{for } \alpha > 0, \]
(3.7) \[ S_{r+s+1}(t) = \frac{\Gamma(r+s+2)}{\Gamma(r+1) \Gamma(s+1)} \int_0^1 S_{r}(ut) \left( \log \frac{1}{u} \right) du \quad \text{for } r > -1, s > -1. \]

Lemma 2.
\[ \frac{2}{\pi} \int_0^\infty S_\alpha(u) \sin xu \, du = \left( \log \frac{1}{x} \right)^\alpha \quad \text{for } 0 < x < 1, \]
\[ = 0 \quad \text{for } x \geq 1. \]

4. Proof of Theorem 1. The \((R, \log, \beta)\) means of the conjugate derived Fourier series is denoted by
\[ R_s(\omega) = \frac{1}{(\log \omega)^s} \sum_{n=-\infty}^{\infty} \left( \log \frac{\omega}{n} \right)^s nA_n(x), \]
and the Fourier series of \(\varphi(t)\) becomes
\[ \varphi(t) \sim \sum_{n=0}^{\infty} A_n(x) \cos nt - s \cos t. \]
Since \(S'_\beta(t)\) and \(S'_{\beta}(t) (\beta \geq 1)\) are integrable in \((0, \infty)\), by Young's theorem, we get
\[ \int_0^\infty S'_\beta(\omega t) \varphi(t) dt = \frac{1}{\omega} \int_0^\infty S'_\beta(\omega t) \cos nt \, t - s \int_0^\infty S'_\beta(\omega t) \cos t \, t, \]
where the \(n\)th term of the right side series is
\[ \int_0^\infty S'_\beta(\omega t) \cos nt \, t = \left[ \frac{1}{\omega} S(\omega t) \cos nt \right]_0^\infty + \frac{n}{\omega} \int_0^\infty S(\omega t) \sin nt \, t \]
\[ = \begin{cases} \frac{\pi}{2} \left( \log \frac{\omega}{n} \right)^s & \text{for } n < 1, \\ 0 & \text{for } n \geq 1, \end{cases} \]
by Lemma 2. Hence
\[ \frac{2}{\pi} \int_0^\infty S'_\beta(\omega t) \varphi(t) \, dt = -\frac{1}{\omega^2} \left( \log \frac{\omega}{1} \right)^s s + \sum_{n=-\infty}^{\infty} A_n(x)n \left( \log \frac{\omega}{n} \right)^s \frac{1}{\omega^s}. \]
Therefore
\[ R_s(\omega) = s = \frac{2}{\pi} \omega^2 \int_0^\infty S'_\beta(\omega t) \varphi(t) \, dt \]
\[ = \frac{2}{\pi} \omega^2 \int_0^\infty \varphi(t) \int_0^\infty \beta S_{\beta-1}(\omega t) - S_{\beta}(\omega t) \, dt, \]
by Lemma 1, (3.6). If we put \(\varphi(t)/t = \xi(t)\), then, by integration by parts, we get
\[ \frac{\omega}{(\log \omega)^s} \int_0^\infty \varphi(t) S_{\beta-1}(\omega t) \, dt \]
\[ = \frac{\omega}{(\log \omega)^s} \left[ \xi(t) S_{\beta-1}(\omega t) \right]_0^\infty - \frac{\omega^2}{(\log \omega)^s} \int_0^\infty \xi(t) S'_{\beta-1}(\omega t) \, dt \]
\[ = R_1 + R_2, \]
where \( p \) is a sufficiently large but fixed number. Assuming \( \beta \geq 2 \), and by (3.2) and \( \int_0^t \varphi(u)/u \, du = O(1) \), we get
\[
R_1 = \frac{\omega}{(\log \omega)^3} \int_0^\infty O \left( \frac{\log \omega e^{\beta-1}}{\omega t} \right) \frac{dt}{t} = O \left( \frac{1}{p} \frac{1}{\log \omega} \right)
\]
and
\[
R_2 = O \left( \frac{\omega^2}{(\log \omega)^3} \int_0^\infty \frac{(\log \omega e^{\beta-1})}{\omega^2 t^2} \, dt \right) = O \left( \frac{1}{p} \frac{1}{\log \omega} \right).
\]
Similarly we get
\[
\omega \int_0^\infty \xi(t) S_\beta(\omega t) \, dt = O(1/p).
\]
On the other hand we have, by integration by parts and by (3.1), (3.6),
\[
\frac{\omega}{(\log \omega)^3} \int_0^\infty \xi(t) S_{\beta-1}(\omega t) \, dt
\]
\[
= -\frac{\omega}{(\log \omega)^3} \left[ g(t) S_{\beta-1}(\omega t) \right]_0^\infty + \frac{\omega}{(\log \omega)^3} (\beta-1) \int_0^\infty g(t) S_{\beta-2}(\omega t) \, dt
\]
\[
= O[g(p)/\log \omega] + o(1) + \frac{\omega(\beta-1)}{(\log \omega)^3} \int_0^\infty g(t) S_{\beta-2}(\omega t) \, dt.
\]
We also get
\[
\frac{\omega}{(\log \omega)^3} \int_0^\infty \xi(t) S_\beta(\omega t) \, dt = O[g(p)] + o(1) + \frac{\omega \beta}{(\log \omega)^3} \int_0^\infty g(t) S_{\beta-1}(\omega t) \, dt.
\]
Summing up above estimations, we see that
\[
(4.2) \quad R_\beta(\omega) - s = \frac{2}{\pi} \frac{\omega}{(\log \omega)^3} \int_0^\infty g(t)[\beta(\beta-1)S_{\beta-2}(\omega t) - \beta S_{\beta-1}(\omega t)] \, dt + o(1),
\]
for sufficiently large \( p \) and \( \beta \geq 2 \).
Suppose \( \beta > 2 \) and let \( h = [\beta - 2] \), then, by \( h \) time application of integration by parts, we get
\[
\int_0^\infty g(t) S_{\beta-2}(\omega t) \, dt = (\beta - 2)(\beta - 3) \cdots (\beta - 2 - h) \int_0^\infty S_{\beta - 2 - h - 1}(\omega t) g_{h+1}(t) \, dt.
\]
Using here the formula
\[
g_{h+1}(t) = \frac{1}{\Gamma(h+1-\beta+2)} \int_t^\infty \left( \frac{u}{t} \right)^{\beta-2} \frac{g_{h+2}(u)}{u} \, du,
\]
and (3.7), we have
\[
(4.3) \quad \int_0^\infty g(t) S_{\beta-2}(\omega t) \, dt
\]
\[
= (\beta-2)(\beta-3) \cdots (\beta - 2 - h) \frac{1}{\Gamma(h+1-\beta+2)} \int_0^\infty S_{\beta - 2 - h - 1}(\omega t) \, dt \int_t^\infty \left( \frac{u}{t} \right)^{\beta-2} \frac{g_{h+2}(u)}{u} \, du
\]
\[
= (\beta-2)(\beta-3) \cdots (\beta - 2 - h) \frac{1}{\Gamma(h+3-\beta)} \int_0^\infty g_{h+2}(u) \, du \int_0^\infty \left( \frac{u}{t} \right)^{\beta-2} S_{\beta - 2 - h - 1}(\omega t) \, dt.
\]
\[
2) \quad \text{Cf. Wang} [6], \text{Lemma 3}.
\]
By similar estimation we get
\[(4.4) \quad \int_0^\infty g(t)S_{\omega t}(\omega t)dt = \Gamma(\beta)\int_0^\infty S_0(\omega t)g_{\beta-1}(t)dt.\]
Substituting (4.3) and (4.4) into (4.2) and using (3.5), we get the following relation
\[(4.5) \quad R_\omega(s) - s = \frac{2}{\pi} \Gamma(\beta + 1) (\log \omega)^s \int_0^\infty \left[ g_{\beta+2}(t) - g_{\beta-1}(t) \right] \frac{1 - \cos \omega t}{t} dt + o(1).\]
Let us now put \(\beta = \alpha + 2 (\alpha \geq 0),\) then
\[(4.6) \quad R_{\alpha+2}(\omega) - s = \frac{2}{\pi} \Gamma(\alpha + 3) (\log \omega)^{s+2} \int_0^\infty \left[ g_\alpha(t) - g_{\alpha+1}(t) \right] \frac{1 - \cos \omega t}{t} dt + o(1).\]
By the assumption of the theorem,
\[g_\alpha(t) = o\left(\left(\log \frac{1}{t}\right)^\alpha\right), \quad g_{\alpha+1}(t) = o\left(\left(\log \frac{1}{t}\right)^{\alpha+1}\right),\]
as \(t \to 0.\) Hence, if we divide the integral (4.6) into those with the ranges \((0, 2/\omega)\) and \((2/\omega, \pi)\) and use the estimation \((1 - \cos \omega t)/t = O(\omega)\) or \(= O(1/t),\) we can easily get
\[(4.7) \quad R_{\alpha+2}(\omega) - s = o(1).\]
Thus Theorem 1 is completely proved.

(To be continued)

References