24. On the Strong Summability of the Derived Fourier Series. II

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1. B. N. Prasad and U. N. Singh [1] have proved the following

Theorem 1. *Let $f(t)$ be a continuous function of bounded variation, with period 2π, and let*

$$g_x(u)=g(u)=f(x+u)-f(x-u)-2us,$$

then, if

$$\int_0^t |dg(u)|=O\left[t\Big/\left(\log\frac{1}{t}\right)^{1+\varepsilon}\right] \qquad (t\to 0) \tag{1}$$

for a positive ε, then the derived Fourier series of $f(t)$ is strongly summable (or H_1-summable) to s at x, that is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}|\tau_\nu(x)-s|=0 \tag{2}$$

$\tau_n(x)$ being the n-th partial sum of the derived Fourier series of $f(x)$.

In the first paper [2], one of us proved that under the assumption of Theorem 1[1)]

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}|\tau_\nu(x)-s|^k=0, \tag{3}$$

for any $k>0$. But in its proof it is used, without stating explicitly, that the summability (H_k) is the local property for the derived Fourier series of $f(x)$. This is true by Wiener's theorem (A. Zygmund [6], p. 221).

We shall now consider an extension of Theorem 1 in the case $k\leqq 2$. In fact we shall prove

Theorem 2. *If*

$$\int_0^t |dg(u)|=O\left[t\Big/\left(\log\frac{1}{t}\right)^{\alpha}\right] \qquad (t\to 0), \tag{4}$$

then

$$\lim\frac{1}{n}\sum_{\nu=1}^{n}|\tau_\nu(x)-s|^2=0 \qquad \textit{for} \qquad \alpha>1/2.$$

This is the analogue of Wang's theorem for Fourier series [3]. We can also prove the following

Theorem 3. *In Theorem 2, if the condition (4) is replaced by*

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1) In [2], $\tau_\nu^*(x)$ may be replaced by $\tau_\nu(x)$ and the last section, containing Theorems 3 and 4, must be omitted.

$$\int_0^t |dg(u)| = o(t), \tag{5}$$

then

$$\sum_{\nu=1}^{n} |\tau_\nu(x) - s|^2 = o(n \log n).$$

This is the analogue of the Hardy-Littlewood theorem for the Fourier series [4] (cf. [5]). We shall omit the proof, since we can prove it by the similar method as Theorem 2.

2. **Proof of Theorem 2.** We can replace O in (4) by o, and then for any ε, there is a δ such that

$$\int_0^t |dg(u)| < \varepsilon t \Big/ \left(\log \frac{1}{t}\right)^\alpha \qquad (0<t<\delta).$$

Let us put

$$g(u) = g_1(u) + g_2(u),$$

where

$$\begin{aligned} g_1(u) &= g(u) \text{ in } (0,\ \delta/2), \\ &= 0 \quad\ \text{ in } (\delta,\ \pi) \end{aligned}$$

and $g_1(u)$ is linear in $(\delta/2, \pi)$ and is continuous in $(0, \pi)$. Hence $g_1(u)$ is also a continuous function of bounded variation which vanishes in the interval $(0, \delta/2)$.

We can easily see that

$$\begin{aligned} \tau_n(x) - s &= \frac{2}{\pi}\int_0^\pi \frac{\sin(n+1/2)t}{2\sin t/2}\, dg(t) \\ &= \frac{2}{\pi}\int_0^\pi \frac{\sin(n+1/2)t}{2\sin t/2}\, dg_1(t) + \frac{2}{\pi}\int_0^\pi \frac{\sin(n+1/2)t}{2\sin t/2}\, dg_2(t) \\ &= \eta_n + \zeta_n, \end{aligned} \tag{6}$$

say. Since ζ_n is n times of the n-th Fourier coefficient of a continuous function of bounded variation, we obtain

$$\frac{1}{n}\sum_{\nu=1}^{n} |\zeta_\nu|^2 = o(1) \tag{7}$$

by the Wiener's theorem.

We have also

$$\begin{aligned} \sum_{\nu=1}^{n} |\eta_\nu|^2 &= \frac{4}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \frac{dg_1(u)}{u}\,\frac{dg_1(v)}{v}\,\frac{\sin n(u-v)}{u-v} + o(n) \\ &= \frac{4}{\pi^2}\left\{\int_{1/n}^{\delta} \frac{dg_1(u)}{u}\int_{u}^{\delta} \frac{dg_1(v)}{v}\,\frac{\sin n(u-v)}{u-v}\right. \\ &\qquad \left. + \int_{1/n}^{\delta} \frac{dg_1(u)}{u}\int_{1/n}^{u} \frac{dg_1(v)}{v}\,\frac{\sin n(u-v)}{u-v}\right\} + o(n) \\ &= \frac{4}{\pi^2}(I_n + J_n) + o(n), \end{aligned}$$

say. Let us now estimate I_n. We write

$$I_n=\int_{1/n}^{\delta}\frac{dg_1(u)}{u}\left\{\int_u^{2u}+\int_{2u}^{\delta}\right\}\frac{dg_1(v)}{v}\,\frac{\sin n(u-v)}{u-v}$$
$$=I_n+I_n^2.$$

Then

$$|I_n^2|\leqq A\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u}\int_{2u}^{\delta}\frac{|dg_1(v)|}{v^2},$$

where the inner integral becomes, by integration by parts,

$$\int_{2u}^{\delta}\frac{|dg_1(v)|}{v^2}=\left[\frac{G(v)}{v^2}\right]_{2u}^{\delta}+2\int_{2u}^{\delta}\frac{G(v)}{v^3}dv$$
$$\leqq\frac{4\varepsilon}{u(\log 1/u)^{\alpha}}+\frac{2\varepsilon}{\delta(\log 1/\delta)^{\alpha}}\leqq\frac{A\varepsilon}{u(\log 1/u)^{\alpha}},$$

where $G(v)=\int_0^v|dg_1(w)|$. Hence

$$|I_n^2|\leqq A\varepsilon\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u^2(\log 1/u)^{\alpha}}$$
$$=A\varepsilon\left[\frac{G(u)}{u^2(\log 1/u)^{\alpha}}\right]_{1/n}^{\delta}+A\varepsilon\int_{1/n}^{\delta}\frac{G(u)}{u^3(\log 1/u)^{\alpha}}du$$
$$\leqq A\varepsilon^2\frac{n}{(\log n)^{2\alpha}}+A\varepsilon^2 n\int_{1/n}^{\delta}\frac{du}{u(\log 1/u)^{2\alpha}}\leqq A\varepsilon^2 n.$$

Concerning I_n^1, we have

$$I_n^1=\int_{1/n}^{\delta}\frac{dg_1(u)}{u}\left\{\int_u^{u+1/n}+\int_{u+1/n}^{2u}\right\}\frac{dg_1(v)}{v}\,\frac{\sin n(u-v)}{u-v}$$
$$=I_n^{1,t}+I_n^{1,2},$$

where

$$|I_n^{1,1}|\leqq n\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u}\int_u^{u+1/n}\frac{|dg_1(v)|}{v}$$
$$\leqq\varepsilon n\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u(\log 1/u)^{\alpha}}\leqq A\varepsilon^2 n\int_{1/n}^{\delta}\frac{du}{u(\log 1/u)^{2\alpha}}\leqq A\varepsilon^2 n$$

and

$$|I_n^{1,2}|\leqq\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u}\int_{u+1/n}^{2u}\frac{|dg_1(v)|}{v(v-u)}.$$

Since $\dfrac{1}{v(v-u)}=\dfrac{1}{u}\left(\dfrac{1}{v-u}-\dfrac{1}{v}\right)$, we obtain

$$|I_n^{1,2}|\leqq\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u}\int_{u+1/n}^{2u}\frac{|dg_1(v)|}{v}+\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u}\int_{u+1/n}^{2u}\frac{|dg_1(v)|}{v-u}$$
$$=I_n^{1,2,1}+I_n^{1,2,2},$$

where

$$I_n^{1,2,1}\leqq I_n^{1,2,2}\leqq A\varepsilon\int_{1/n}^{\delta}\frac{|dg_1(u)|}{u^2}\,\frac{nu}{(\log 1/u)^{\alpha}}\leqq A\varepsilon^2 n.$$

Thus we have proved that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}|\eta_\nu|^2\leqq A\varepsilon^2.$$

By (6) and (7)

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}|\tau_\nu(x)-s|^2\leqq A\varepsilon.$$

Since ε is arbitrary, we get the required result.

We have first proved Theorem 2 for the case $k=1$. By the remark of T. Tsuchikura, we have gotten the case $k\leqq 2$. We express him our hearty thanks.

References

[1] B. N. Prasad and U. N. Singh: On the strong summability of the derived Fourier series and its conjugate series, Math. Zeits., **56**, 280–288 (1952).
[2] M. Kinukawa: On the strong summability of the derived Fourier series, Proc. Japan Acad., **30**, 801–804 (1954).
[3] F. T. Wang: Note on H_2 summability of Fourier series, Jour. London Math. Soc., **19**, 208–209 (1944).
[4] G. H. Hardy and J. E. Littlewood: The strong summability of Fourier series, Fund. Math., **25**, 162–189 (1935).
[5] T. Kawata: A proof of a theorem of Hardy and Littlewood concerning strong summability of Fourier series, Proc. Imp. Acad., **15**, 243–246 (1939).
[6] A. Zygmund: Trigonometrical series (1935).