52. On Homotopy Groups of Function Spaces

By Yoshiro INOUE

(Comm. by K. KUNUGI, M.J.A., April 12, 1955)

1. Introduction: Since M. Abe defined "Abe groups" [1], various kinds of homotopy groups of function spaces have been introduced by some authors. These groups are considered as homotopy groups of suitable pseudo fibre spaces. Our purpose of this paper is to investigate the homotopy groups of J. R. Jackson [6] and the abhomotopy groups introduced by S. T. Hu [3] from this point of view.

2. Pseudo fibre spaces: By a pseudo fibre space (X, p, B), we understand a collection of two spaces X, B and a continuous mapping $p: X \longrightarrow B$ of X onto B satisfying the "Lifting homotopy theorem" (p. 63, P. J. Hilton [2]; p. 443, J. P. Serre [7]). We assume that X is arcwise connected. The projection $p: X \longrightarrow B$ induces the isomorphism:

(1) $$p'_n: \pi_n(X, X_0) \longrightarrow \pi_n(B), \qquad n \geq 2,$$

and the homomorphism: $p_n: \pi_n(X) \longrightarrow \pi_n(B)$, $n \geq 1$, where X_0 is the fibre over a point $b_0 \in B$. It is well known that the homotopy sequence:

(2) $$\cdots \longrightarrow \pi_{n+1}(B) \xrightarrow{d_{n+1}} \pi_n(X_0) \xrightarrow{i_n} \pi_n(X) \xrightarrow{p_n} \pi_n(B) \longrightarrow \cdots, \qquad n \geq 1,$$

of (X, p, B) is exact. The main results of this paper will be based on the following two theorems.

Theorem 1. *If the pseudo fibre space (X, p, B) admits a cross section, then we have the direct sum relation*

$$\pi_n(X) \approx \pi_n(X_0) + \pi_n(B), \qquad n \geq 2,$$

and $\pi_1(X)$ contains two subgroups M and N such that M is normal and isomorphic to $\pi_1(X_0)$, p_1 maps N isomorphically onto $\pi_1(B)$ and each element of $\pi_1(X)$ is uniquely representable as the product of an element of M with an element of N. (For example, see Theorem 27.6; S. T. Hu [4].)

Theorem 2. *Let (X, p, B) be a pseudo fibre space. If the total space X is deformable into the fibre X_0 relative to a point $x_0 \in X_0$, then we have the direct sum relation (direct product, for $n=1$):*

$$\pi_n(X_0) \approx \pi_{n+1}(B) + \pi_n(X), \qquad n \geq 1.$$

(Proof) For $n \geq 2$, the theorem follows from Theorem 27.10, S. T. Hu [4]. According to the same theorem, $\pi_1(X_0)$ contains two subgroups M and N such that d_2 maps $\pi_2(B)$ isomorphically onto M, i_1 maps N isomorphically onto $\pi_1(X)$ and each element of $\pi_1(X_0)$ is uniquely representable as the product of an element of M and an element of N. Thus the proof is complete, if we prove that N is normal. This fact follows at once from the following theorem.

Theorem 3. *Let (X, p, B) be a pseudo fibre space. Let ξ be an arbitrary element of $\pi_2(B)$. Then the element $d_2\xi \in \pi_1(X_0)$ induces the identical automorphism of $\pi_1(X_0)$ in the sense of S. Eilenberg for every integer $n \geq 1$, where X_0 is the fibre over a point $b_0 \in B$.*

(Proof) First of all, we recall that $\pi_2(B) \approx \pi_2(X, X_0)$. Then, for ξ, there exists a map $\omega : I^2, I^1, J \longrightarrow X, X_0, x_0$ such that $p\omega$ represents ξ and $\omega \mid I^1$ represents the element $d_2\xi$. Let f be a map of an element α of $\pi_n(X_0)$. From the definition, there exists a homotopy $h_t : I^n \longrightarrow X_0$ such that $h_0 = f$, $h_t(\dot{I}^n) = \omega(1-t, 0)$, and h_1 represents the element $(d_2\xi)^*\alpha$, where $(d_2\xi)^*$ is the operator of $\pi_n(X_0)$ induced by $d_2\xi$. Define a map $F : Q = \dot{I}^n \times I \times I \smile I^n \times I \times \dot{I} \smile I^n \times 0 \times I \longrightarrow X$ by taking from each $(x^n, t, s) \in Q$

$$F(x^n, t, s) = \begin{cases} f(x^n) & \text{on } I^n \times I \times 1 \smile I^n \times 0 \times I \\ \omega(1-t, s) & \text{on } \dot{I}^n \times I \times I \\ h_t(x^n) & \text{on } I^n \times I \times 0. \end{cases}$$

The map $pF : Q \longrightarrow B$ is extended to the map $G' : I^n \times I \times I \longrightarrow B$ such that $G'(x^n, t, s) = p\omega(1-t, s)$. Then, by Proposition 1, p. 443, J. P. Serre [7], the map F is extended to the map $F' : I^n \times I \times I \longrightarrow X$ such that $pF' = G'$. The homotopy $H_t : I^n \longrightarrow X$ defined by $H_t(x^n) = F'(x^n, 1, t)$ is a homotopy joining the map $f = H_1$ and the map $h_1 = H_0$. This completes the proof.

In the above theorem, if the boundary homomorphism $d_2 : \pi_2(B) \longrightarrow \pi_1(X_0)$ is onto, the fibre X_0 is n-simple for every integer $n \geq 1$. Especially, if the fibre X_0 is contractible in X to a point $x_0 \in X_0$ relative to x_0, X_0 is n-simple for any integer $n \geq 1$.

3. Function spaces: (i) Let Y be a given space. We say that a space X belongs to the class $\mathfrak{A}(Y)$ provided that whenever $\sigma : T \longrightarrow Y^X$ is a continuous mapping of an arbitrary finite simplicial complex T to Y^X, we may define a continuous mapping $\sigma' : T \times X \longrightarrow Y$ by

$$\sigma'(t, x) = \sigma(t)(x) \qquad (t, x) \in T \times X,$$

where Y^X is a function space of compact open topology consisting of all maps $f : X \longrightarrow Y$. For example, if X is locally compact and regular, X belongs to the class $\mathfrak{A}(Y)$ for any space Y. If X satisfies the first axiom of countability, X belongs to $\mathfrak{A}(Y)$ for any space Y.

Let (X, X_0) be a pair of spaces X, X_0 such that X_0 is a closed subset of X. If, for any finite simplicial complex T, $T \times X_0$ has the homotopy extention property in $T \times X$ with respect to Y, we say that the pair (X, X_0) belongs to the class $\mathfrak{B}(Y)$. If X and X_0 are ANR's, (X, X_0) belongs to $\mathfrak{B}(Y)$ for any space Y. Especially, if X_0 is a subcomplex of a finite simplicial complex X, (X, X_0) belongs to

$\mathfrak{B}(Y)$ for any space Y.

Theorem 4. *For a pair (X, X_0), the following conditions are equivalent.*

(1) (X, X_0) *belongs to the class* $\mathfrak{B}(Y)$.

(2) *A map* $f:(K\times X_0)\smile(L\times X)\longrightarrow Y$ *has an extention* $F:K\times X\longrightarrow Y$ *for any pair* (K, L) *such that* L *is a subcomplex of a finite simplicial complex* K *and* K, L *are contractible in itself.*

(3) *The set* $(I^m\times X_0)\smile(\dot{I}^m\times X)$ *has the homotopy extention property in* $I^m\times X$ *with respect to* Y, $m\geq 0$ (see § 4, J. R. Jackson [6]).

(Proof) (1)$\longrightarrow$(2): Refer to the proof of Proposition 1, p. 443, J. P. Serre [7]. (2)$\longrightarrow$(3), (3)$\longrightarrow$(1): It is clear.

(ii) Let $(X; X_1, X_2, X_3)$, $(Y; Y_1, Y_2, Y_3)$ be two tetrads such that X_1, X_2, X_3 are closed subsets of X, and $X_3\subseteq X_1\frown X_2$, $Y_3\subseteq Y_1\frown Y_2$. Denote by Ω the function space of compact open topology consisting of all maps $f:(X; X_1, X_2, X_3)\longrightarrow(Y; Y_1, Y_2, Y_3)$ and by $\mathfrak{B}$ the function space of compact open topology consisting of all maps $f:(X_1, X_1\frown X_2, X_3)\longrightarrow(Y_1, Y_1\frown Y_2, Y_3)$. In the remainder of this paper, we shall always consider the spaces Ω, $\mathfrak{B}$ under the following conditions.

(I) *X and X_1 belong to classes $\mathfrak{A}(Y)$ and $\mathfrak{A}(Y_1)$ respectively.*

(II) *$(X, X_1\smile X_2)$ and $(X_2, X_1\frown X_2)$ belong to $\mathfrak{B}(Y)$ and $\mathfrak{B}(Y_2)$ respectively.*

Define a continuous mapping $p:\Omega\longrightarrow\mathfrak{B}$ by making correspondence $f\in\Omega$ to the partial map $pf=f\mid X_1\in\mathfrak{B}$. Let f be an arbitrary map of Ω, and let g be a map of $\mathfrak{B}$ such that g is joined by arc in $\mathfrak{B}$ with pf. Then there exists a homotopy $h_t:(X_1, X_1\frown X_2, X_3)\longrightarrow(Y_1, Y_1\frown Y_2, Y_3)$ such that $h_0=pf$, $h_1=g$. Since $(X_2, X_1\frown X_2)$ belongs to $\mathfrak{B}(Y_2)$, there exists an extention $h'_t:(X_2, X_1\frown X_2, X_3)\longrightarrow(Y_2, Y_1\frown Y_2, Y_3)$ of $h_t\mid X_1\frown X_2$ such that $h'_0=f\mid X_2$. Define a homotopy $H_t:(X_1\smile X_2, X_1, X_2, X_3)\longrightarrow(Y_1\smile Y_2, Y_1, Y_2, Y_3)$ by

$$H_t\mid X_1=h_t, \qquad H_t\mid X_2=h'_t.$$

Since $(X, X_1\smile X_2)$ belongs to $\mathfrak{B}(Y)$, H_t has an extention $H'_t:(X, X_1, X_2, X_3)\longrightarrow(Y, Y_1, Y_2, Y_3)$ such that $H'_0=f$. Then the map $H'_1:(X, X_1, X_2, X_3)\longrightarrow(Y, Y_1, Y_2, Y_3)$ belongs to Ω and $pH'_1=g$. Denote by $\Omega(f)$ the arcwise connected component of Ω containing the map $f\in\Omega$. Then we have the following lemma from the above arguments.

Lemma 5. *The partial map $p\mid\Omega(f)$ maps $\Omega(f)$ onto $\mathfrak{B}(pf)$.*

The following theorem is a main theorem in this paper.

Theorem 6. *$(\Omega(f), p, \mathfrak{B}(pf))$ is a pseudo fibre space.*

(Proof) Let K be an arbitrary finite simplicial complex. Let $F:K\longrightarrow\Omega(f)$ be a map such that $G=pF:K\longrightarrow\mathfrak{B}(pf)$ admits a homotopy $G_t:K\longrightarrow\mathfrak{B}(pf)$. Define a homotopy $G'_t:(K\times X_1, K\times(X_1\frown X_2), K\times X_3)\longrightarrow(Y_1, Y_1\frown Y_2, Y_3)$ by taking for each $s\in K$, $x\in X$

$$G'_t(s, x) = G_t(s)(x).$$

As in the proof of Lemma 5, $G'_t(s, x)$ has an extention $F'_t(s, x) : (K \times X, K \times X_1, K \times X_2, K \times X_3) \longrightarrow (Y, Y_1, Y_2, Y_3)$ such that $F'_0(s, x) = F(s)(x)$. Define a homotopy $F_t : K \longrightarrow \Omega(f)$ by $F_t(s)(x) = F'_t(s, x)$. Since $pF_t = G_t$, $F_0 = F$, the homotopy is a desired homotopy.

By this theorem, $(\Omega, p, p\Omega)$ is a pseudo fibre space. Then, we have the isomorphism: $\pi_n(\Omega, \Phi, f) \approx \pi_n(\mathfrak{B}, pf)$, $n \geq 2$, and the homotopy sequence:

$$\longrightarrow \pi_{n+1}(\mathfrak{B}, pf) \xrightarrow{d_{n+1}} \pi_n(\Phi, f) \xrightarrow{i_n} \pi_n(\Omega, f) \xrightarrow{p_n} \pi_n(\mathfrak{B}, pf) \longrightarrow \cdots, \quad n \geq 1,$$

where Φ is the fibre over pf. In the remainder of this paper, we always consider the homotopy sequence above when f is the constant map $k_{y_0} : X \longrightarrow y_0$ of X onto a single point of Y. In this case, the fibre Φ_0 over pk_{y_0} is the function space $Y^X\{X_2, X_1; Y_2, y_0\}$ consisting of all maps $f : (X, X_2, X_1) \longrightarrow (Y, Y_2, y_0)$.

4. Results of Jackson: In this section, we shall investigate the pseudo fibre space $(\Omega, p, p\Omega)$ defined in the preceding section under the following condition:

(CI) *There exists a retraction* $\omega : X \longrightarrow X_1$ *such that the partial map* $\omega \mid X_2$ *maps* X_2 *onto* $X_1 \frown X_2$.

Theorem 7. *If* $(X; X_1, X_2)$ *satisfies the condition* (CI), *the pseudo fibre space* $(\Omega, p, p\Omega)$ *has a cross section.*

(Proof) Define a continuous mapping $\psi : p\Omega \longrightarrow \Omega$ by

$$(\psi f)(x) = f(\omega(x)), \quad x \in X, \ f \in p\Omega.$$

Then $p\psi f = f$, and ψ is a cross section.

By this theorem, under the condition (CI), Theorem 1 is applicable.

Theorem 8. *Under the condition (CI), the direct sum relation:*

$$\pi_n(\Omega, k_{y_0}) \approx \pi_n(\Phi_0, k_{y_0}) + \pi_n(\mathfrak{B}, k_{y_0}), \quad n \geq 2,$$

holds, where k_{y_0} *is the constant map:* $X \longrightarrow y_0$ *or* $X_1 \longrightarrow y_0$. $\pi_1(\Omega, k_{y_0})$ *contains a normal subgroup* M *isomorphic to* $\pi_1(\Phi_0, k_{y_0})$ *and a subgroup* N *isomorphic to* $\pi_1(\mathfrak{B}, k_{y_0})$ *and each element of* $\pi_1(\Omega, k_{y_0})$ *is uniquely representable as the product of an element of* M *with an element of* N.

When the spaces X_1, X_2, X_3; Y_1, Y_2, Y_3 satisfy the following conditions respectively, Theorems 7 and 8 hold.

(i) $X_1 = X_2 = X_3$, $Y_1 = Y_2 = Y_3$, *and* X_1 *is a retract of* X (Theorem (10.2); J. R. Jackson [6]).

(ii) $X_2 = X_3 =$ *a single point* x_0, $Y_2 = Y_3 =$ *a single point* y_0, *and* X_1 *is a retract of* X (Theorem (10.3); [6]).

(iii) $X_1 =$ *a single point* x_0.

In the cases (i) and (ii), if X_1 is a deformation retract of X, then $\pi_n(\Omega, k_{y_0}) \approx \pi_n(\mathfrak{B}, k_{y_0})$, $n \geq 1$ (Theorem (8.1); J. R. Jackson [6]). In fact, the space Φ_0 is contractible to a point, then $\pi_n(\Phi_0, k_{y_0}) = 0$, $n \geq 1$.

Example 1. Denote by s^p an arbitrary but fixed point of p-sphere S^p. Let $X=S^p\vee S^q$ be the union of S^p and S^q joined together by identifying the point s^p and s^q to a single point x_0. Consider the spaces:

$$\Omega=Y^{S^p\vee S^q}\{S^p, S^q, x_0; Y_1, Y_2, y_0\}, \qquad \mathfrak{B}=Y_1^{S^p}\{s^p, y_0\}.$$

The triple $(\Omega, p, \mathfrak{B})$ is a pseudo fibre space, and there exists a retraction ω of $S^p\vee S^q$ onto S^p satisfying the condition (CI). The fibre Φ_0 over the constant map k_{y_0} is homeomorphic to the space $Y_2^{S^q}\{s^q, y_0\}$. Then, from the well-known relations: $\pi_m(Y_2^{S^q}\{s^q, y_0\}, k_{y_0})\approx\pi_{m+q}(Y_2, y_0)$, $\pi_m(Y_1^{S^p}\{s^p, y_0\}, k_{y_0})\approx\pi_{m+p}(Y_1, y_0)$, the direct sum relation:

$$\pi_m(\Omega, k_{y_0})\approx\pi_{m+p}(Y_1, y_0)+\pi_{m+q}(Y_2, y_0), \qquad m\geq 1,$$

follows at once from Theorem 8 (cf. Theorem (13.3); [6]).

Example 2. By putting $X=S^p$, $X_1=X_2=X_3=s^p$, $Y=Y_1=Y_2=Y_3$, the spaces Ω and $\mathfrak{B}$ are homeomorphic to the spaces Y^{S^p} and Y respectively. The pseudo fibre space (Y^{S^p}, p, Y) has a cross section as the case (iii). From the relation: $\kappa^n_{r-1}(Y, y_0)\approx\pi_r(Y^{S^{n-r}}, k_{y_0})$, $0<r\leq n$, of S. T. Hu ((5.2); S. T. Hu [3]) and Theorem 8, we have the direct sum relation:

$$\kappa^{m+p}_{m-1}(Y, y_0)\approx\pi_{m+p}(Y, y_0)+\pi_m(Y, y_0), \quad m\geq 2,\ p\geq 0,$$

of the $(m+p, m-1)$-th abhomotopy group of Y. (For the Abe group $\kappa_0^{1+p}(Y, y_0)\approx\pi_1(Y^{S^p}, k_{y_0})$, see the paper [5].)

5. A generalization of abhomotopy groups: In this section, we shall investigate the pseudo fibre space $(\Omega, p, p\Omega)$ defined by

$$\Omega=Y^X\{X_1, X_2, x_0; Y_1, Y_1, y_0\}, \quad p\Omega\subseteq\mathfrak{B}=Y_1^{X_1}\{x_0, y_0\}$$

under the conditions:

(CII, 1) $X_1\subseteq X_2$,

(CII, 2) X_1 *has the homotopy extention property in* X_2 *with respect to* X_2,

(CII, 3) X_2 *has the homotopy extension property in* X *with respect to* X,

(CII, 4) X_1 *is contractible in* X_2 *to a point* x_0 *relative to* x_0.

The fibre Φ_0 over k_{y_0} is the function space $Y^X\{X_2, X_1; Y_1, y_0\}$.

Lemma 9. *Under the conditions* (CII, 1) — (CII, 4), Ω *is deformable into the fibre* Φ_0 *relative to* k_{y_0}.

(Proof) Let $\omega_t: X_1\longrightarrow X_2$ be a homotopy such that $\omega_0(x)=x (x\in X_1)$, $\omega_t(x_0)=x_0\,(0\leq t\leq 1)$, $\omega_1(X_1)=x_0$. From conditions (CII, 2) and (CII, 3), ω_t has an extension $\omega'_t:(X, X_2)\longrightarrow(X, X_2)$ such that $\omega'_0(x)=x$, $x\in X$. Define a homotopy $W_t:\Omega\longrightarrow\Omega$ by

$$(W_tf)(x)=f(\omega'_t(x)), \quad x\in X.$$

Clearly, $W_0f=f$ and $W_1f\in\Phi_0$. This completes the proof.

By this lemma and Theorem 2, we have the following theorem.

Theorem 10. *Under the conditions* (CII, 1) — (CII, 4), *the direct sum relation* (*direct product for* $n=1$):

$\pi_n(Y^X\{X_2,X_1;Y_1,y_0\},k_{y_0})\approx\pi_{n+1}(Y_1^{X_1}\{x_0,y_0\},k_{y_0})+\pi_n(Y^X\{X_2,x_0;Y_1,y_0\},k_{y_0})$ *holds for any integer* $n\geq 1$.

Example 1. When $X=p$-sphere S^p, $p\geq 1$,

$X_1=r$-th subcomplex K^r of S^n, $p\geq r\geq 0$,

$X_2=X$, $x_0=$ a single point of K^r, $Y_1=Y$,

we have the direct sum relation:

$$\pi_n(Y^{S^p}\{K^r,y_0\},k_{y_0})\approx\pi_{n+1}(Y^{K^r}\{x_0,y_0\},k_{y_0})+\pi_{n+p}(Y,y_0),\quad n\geq 1.$$

Here we recall that $\pi_n(Y^{S^p}\{x_0,y_0\},k_{y_0})\approx\pi_{n+p}(Y,y_0)$. Especially, by putting $K^r=S^r$, we have the direct sum relation of the abhomotopy group $\kappa_{n+r}^{n+p}(Y,y_0)$:

$$\kappa_{n+r}^{n+p}(Y,y_0)\approx\pi_n(Y^{S^p}\{S^r,y_0\},k_{y_0})\approx\pi_{n+r+1}(Y,y_0)+\pi_{n+p}(Y,y_0)$$

(§5; S. T. Hu [3]). It is easily seen that the group $\pi_1(Y^{S^n}\{K^r,y_0\},k_{y_0})$ is the group $\sigma^{(n+1,r+1)}(Y,y_0)$, $r\geq 0$, of H. Uehara [8].

Example 2. When $X=p$-cell E^p, $p\geq 2$,

$X_2=S^{p-1}=$ the boundary of E^p,

$X_1=S^i=i$-th sphere in S^{p-1}, $i\leq p-2$,

$x_0=$ a single point of S^i,

we have the direct sum relation:

$$(*)\qquad \begin{aligned}\pi_n(Y^{E^p}\{S^{p-1},S^i;Y_1,y_0\},k_{y_0})&\approx\pi_{n+1}(Y_1^{S^i}\{x_0,y_0\},k_{y_0})\\&\quad+\pi_n(Y^{E^p}\{S^{p-1},x_0;Y_1,y_0\},k_{y_0})\\&\approx\pi_{n+i+1}(Y_1,y_0)+\pi_{n+p}(Y,Y_1,y_0).\end{aligned}$$

One can easily be seen that these groups are relativized groups of abhomotopy groups. Following to S. T. Hu, one can define the group $\pi_n(Y^{E^p}\{S^{p-1},S^i;Y_1,y_0\},k_{y_0})$ directly and give its direct sum relation (*) by the same arguments as in the paper [3]. For an another definition of relativized abhomotopy groups, see the paper [5].

References

[1] M. Abe: Über die stetigen Abbildungen der n-Sphäre in einen metrischen Raum, Jap. J. Math., **16**, 169–176 (1940).

[2] P. J. Hilton: Homotopy Theory, Victoria University (1952).

[3] S. T. Hu: On spherical mapping in a metric spaces, Ann. Math., **48**, 717–734 (1947).

[4] ——: Homotopy Theory, **1**, Tulane University (1950).

[5] Y. Inoue: Some remarks on abhomotopy groups, Proc. Japan Acad., **31**, 60–65 (1955).

[6] J. R. Jackson: On homotopy groups of function spaces, Amer. J. Math., **74**, 241–252 (1952).

[7] J. P. Serre: Homology singulière des espaces fibrés, Ann. Math., **54**, 425–505 (1951).

[8] H. Uehara: On a generalization of the Abe groups, Jap. J. Math., **2**, 231–246 (1951).