51. On the Property of Lebesgue in Uniform Spaces

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In this Note, we shall introduce a new concept, the property of Lebesgue in a uniform space.*) Some properties of Lebesgue property in metric spaces were studied in 1950 by A. A. Monteiro and M. M. Peixoto ((2), (3)).

Let S be a topological space. A covering of S is a family of open sets whose union is S. The covering is called *binary* if it consists of two open sets or *finite* if it consists of a finite family of open sets.

Now we shall consider a separated uniform space E. Let $\mathfrak{S}$ be the filter of surroundings of E. If $A\subset E$, and $V\in\mathfrak{S}$, we denote by $V(A)$ the image of the set $(E\times A)\frown V$ by the projection of $E\times E$ onto the first factor E.

We say that a covering $\mathfrak{F}=\{O_\alpha\}$ of E has the *Lebesgue property* if there is a surrounding V of $\mathfrak{S}$ such that, for each x of E, we can find an open set O_α satisfying $V(x)\subset O_\alpha$.

It is clear that, if any finite covering has the Lebesgue property, so is binary covering. We shall prove the following

Theorem 1. If a uniform space induced by $\mathfrak{S}$ is normal and every binary covering has the Lebesgue property, then every finite covering has the Lebesgue property.

Proof. Let $O_i (i=1, 2, \dots, n)$ be a finite covering of E. By the normality of E, we can find a covering $\{G_i\}$ such that $G_i\subset \overline{G}_i\subset O_i$. Therefore $\bigcup_{i=1}^n G_i=\bigcup_{i=1}^n \overline{G}_i=E$. Let $H_i=E-\overline{G}_i$, then, for each i, $\{O_i, H_i\}$ is a binary covering of E, and it has the Lebesgue property. Let V_i be a surrounding for the covering $\{O_i, H_i\}$, and put $V=\bigcap_{i=1}^n V_i$, then $V(x)\subset V_i(x)$ $(i=1, 2, \dots, n)$ for every x of E. Suppose that $V(x)\subset H_i$ $(i=1, 2, \dots, n)$, then

$$V(x)\subset \bigcap_{i=1}^n H_i=\bigcap_{i=1}^n (E-\overline{G}_i)=E-\bigcup_{i=1}^n \overline{G}_i=\text{empty}.$$

Hence there is at least one of i such that $V(x)\subset O_i$. Q.E.D.

To prove that any compact space has the Lebesgue property, we shall show the following

Theorem 2. The following two properties are equivalent:

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*) Throughout this Note, we use the basic concepts of uniform spaces in N. Bourbaki (1). We shall use the terminologies in P. Samuel (4).

(1) *Any binary covering of a uniform space E has the Lebesgue property.*

(2) *For every pair of disjoint closed sets F_1, F_2, there is a surrounding V such that $V(F_1)\frown F_2=F_1\frown V(F_2)=O$.*

Proof. (1) $\rightarrow$ (2). Let $O_i=E-F_i$ $(i=1,2)$, then $\{O_1, O_2\}$ is a binary covering of E. Since the covering $\{O_1, O_2\}$ has the Lebesgue property, there is a surrounding V such that $V(x)\subset O_1$ or $V(x)\subset O_2$ for any x of E. If $x\in F_1$, then $x\bar{\in} O_1$, hence $V(x)\subset O_2$. This shows that $V(F_1)\frown F_2=O$. Similarly we have $F_1\frown V(F_2)=O$.

(2) $\rightarrow$ (1). Let $\mathfrak{F}=\{O_1, O_2\}$ be a binary covering of E. If $E=O_1$ or $E=O_2$, the theorem is trivial. Let $E\neq O_i$ $(i=1, 2)$, then two sets $F_1=E-O_1$, $F_2=E-O_2$ are non-empty, disjoint and closed. Therefore, there is a symmetric surrounding V such that $V(F_1)\frown F_2=F_1\frown V(F_2)=O$. Let U be a symmetric surrounding such that $U\circ U\subset V$. Then we show $U(x)\subset O_1$ or $U(x)\subset O_2$. Suppose that there is an element a such that $U(a)\not\subset O_1$ and $U(a)\not\subset O_2$. Therefore, there are $x_1\bar{\in} O_1$, $x_2\bar{\in} O_2$ such that $(a, x_1)\in U$ and $(a, x_2)\in U$. Since $x_1\in F_1$, $x_2\in F_2$, and $(x_1, x_2)\in U\circ U\subset V$, we have $V(F_1)\frown F_2\neq O$, which is contradiction. Hence the equivalence is proved. Q.E.D.

From Theorems 1, 2, and a well-known theorem (see N. Bourbaki (1), Chap. 2, p. 162), we have

Theorem 3. If a uniform space is compact, it has the Lebesgue property.

Theorem 3 is a generalisation of a theorem by A. A. Monteiro and M. M. Peixoto ((3), p. 112). The relation between the Lebesgue property and uniformly continuity will appear in a later paper.

References

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4) P. Samuel: Ultrafilters and compactification of uniform spaces, Trans. Amer. Math. Soc., **64**, 100–132 (1948).