47. On the Riesz Logarithmic Summability of the Conjugate Derived Fourier Series. II[1]

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5. Proof of Theorem 2. We shall consider the integral

$$I_1=\frac{1}{(\log\omega)^{\alpha+1}}\int_0^\pi g_\alpha(t)\frac{1-\cos\omega t}{t}dt,\quad(\alpha\geqq 0),$$
$$=\frac{1}{(\log\omega)^{\alpha+1}}\left\{\int_0^{\pi/\omega}+\int_{\pi/\omega}^\pi\right\}=I_{1,1}+I_{1,2},$$

say. Integrating by parts, we have

$$I_{1,1}=\frac{1}{(\log\omega)^{\alpha+1}}\left[g_\alpha^1(t)\frac{1-\cos\omega t}{t}\right]_0^{\pi/\omega}$$
$$-\frac{1}{(\log\omega)^{\alpha+1}}\int_0^{\pi/\omega}g_\alpha^1(t)\frac{t\omega\sin\omega t-(1-\cos\omega t)}{t^2}dt$$
$$=o\left[\frac{1}{(\log\omega)^{\alpha+1}}(\log\omega)^\alpha\right]+o\left[\frac{\omega^2}{(\log\omega)^{\alpha+1}}\int_0^{\pi/\omega}t\left(\log\frac{1}{t}\right)^\alpha dt\right]$$
$$=o(1/\log\omega)=o(1),$$

since $g_\alpha^1(t)=o[t(\log 1/t)^\alpha]$ by the assumption of Theorem 2. Also

$$I_{1,2}=\frac{1}{(\log\omega)^{\alpha+1}}\int_{\pi/\omega}^\pi\frac{g_\alpha(t)}{t}dt-\frac{1}{(\log\omega)^{\alpha+1}}\int_{\pi/\omega}^\pi\frac{g(t)}{t}\cos\omega t\,dt$$
$$=I_{1,2,1}-I_{1,2,2},$$

say, where

$$I_{1,2,1}=\frac{1}{(\log\omega)^{\alpha+1}}\left[\frac{g_\alpha^1(t)}{t}\right]_{\pi/\omega}^\pi+\frac{1}{(\log\omega)^{\alpha+1}}\int_{\pi/\omega}^\pi g_\alpha^1(t)\frac{1}{t^2}dt=o(1)$$

and

$$2(\log\omega)^{\alpha+1}I_{1,2,2}=2\int_{\pi/\omega}^\pi g_\alpha(t)\frac{\cos\omega t}{t}dt$$
$$=\int_{\pi/\omega}^{2\pi/\omega}g_\alpha(t)\frac{\cos\omega t}{t}dt+\int_\pi^{\pi+\pi/\omega}g_\alpha(t)\frac{\cos\omega t}{t}dt$$
$$+\int_{\pi/\omega}^\pi\left\{\frac{g_\alpha(t)}{t}-\frac{g_\alpha(t+\pi/\omega)}{t+\pi/\omega}\right\}\cos\omega t\,dt.$$

The first term of the above expression is $o[(\log\omega)^{\alpha+1}]$, as in the estimation of $I_{1,1}$ and the second term is $o(1)$, as easily may be seen. On the other hand, the third term becomes

$$\int_{\pi/\omega}^\pi\frac{g_\alpha(t)-g(t+\pi/\omega)}{t}\cos\omega t\,dt$$

1) Continued from p. 125. References are cited on p. 125.

$$+\int_{\pi/\omega}^{\pi} g_\alpha(t+\pi/\omega)\left\{\frac{1}{t}-\frac{1}{t+\pi/\omega}\right\}\cos\omega t\,dt$$

$$=o\left[(\log\omega)^{\alpha+2}\right]+\frac{\pi}{\omega}\int_{\pi/\omega}^{\pi} g_\alpha(t+\pi/\omega)\frac{\cos\omega t}{t(t+\pi/\omega)}dt$$

$$=o\left[(\log\omega)^{\alpha+2}\right]+\frac{\pi}{\omega}\left[g_\alpha^1(t+\pi/\omega)\frac{\cos\omega t}{t(t+\pi/\omega)}\right]_{\pi/\omega}^{\pi}$$

$$+\int_{\pi/\omega}^{\pi} g_\alpha^1(t+\pi/\omega)\left\{\frac{1}{t^2}-\frac{1}{(t+\pi/\omega)^2}\right\}\cos\omega t\,dt$$

$$-\frac{\pi}{\omega}\int_{\pi/\omega}^{\pi} g_\alpha^1(t+\pi/\omega)\frac{\omega\sin\omega t}{t(t+\pi/\omega)}dt=o\left[(\log\omega)^{\alpha+2}\right].$$

Collecting the above estimations, we find $I_{1,2,2}=o(\log\omega)$. Hence we get $I_{1,2}=o(\log\omega)$. Thus we have

$$\frac{1}{(\log\omega)^{\alpha+2}}\int_0^{\pi} g_\alpha(t)\frac{1-\cos\omega t}{t}dt=o(1),\ \text{as}\ \omega\to\infty. \tag{5.1}$$

Integrating by parts and using the assumption of Theorem 2, we get $g_{\alpha+1}(t)=o[(\log 1/t)^{\alpha+1}]$, and hence, as in the estimation of (4.6), we can see

$$\frac{1}{(\log\omega)^{\alpha+2}}\int_0^{\pi} g_{\alpha+1}(t)\frac{1-\cos\omega t}{t}dt=o(1). \tag{5.2}$$

Thus, by (4.6), (5.1), and (5.2), we get Theorem 2.

6. Proof of Theorem 3. We require some lemmas.

Lemma 3.[2] *If* $g_\alpha^{1+\delta'}(t)=o[t^{1+\delta'}(\log 1/t)^\alpha]$, *for* $\alpha>0$ *and* $\delta>\delta'>0$, *then* $g_{\alpha+1+\delta}(t)=o[(\log 1/t)^{\alpha+1+\delta}]$, *as* $t\to 0$.

Lemma 4.[3] *If we suppose that* $g_\alpha^1(t)=o[t(\log 1/t)^\beta]$, *then* $g_\alpha^{1+\delta}(t)=o[t^{1+\delta}(\log 1/t)^\beta]$, *where* $\delta>0$ *and* $\beta\geqq 0$.

Lemma 5.[4] *For* $\alpha\geqq 0$, $\delta>0$, *we have*

$$g_\alpha^{1+\delta}(t)-g_{\alpha+1}^{1+\delta}(t)=\delta g_{\alpha+1}^{1+\delta}(t)-tg_{\alpha+1}^{\delta}(t).$$

We shall now prove Theorem 3. By the assumption of the theorem and by the formula (4.5), we get

$$\int_0^{\pi}\left[g_{\alpha-2}(t)-g_{\alpha-1}(t)\right]\frac{1-\cos\omega t}{t}dt=o\left[(\log\omega)^\alpha\right].$$

However, by integration by parts,

$$\int_0^{\pi}\left[g_{\alpha-2}(t)-g_{\alpha-1}(t)\right]\frac{1-\cos\omega t}{t}\,dt$$

$$=-\left[\left\{g_{\alpha-1}(t)-g_\alpha(t)\right\}(1-\cos\omega t)\right]_0^{\pi}+\omega\int_0^{\pi}\left[g_{\alpha-1}(t)-g_\alpha(t)\right]\sin\omega t\,dt$$

$$=\omega\int_0^{\pi}\left[g_{\alpha-1}(t)-g_\alpha(t)\right]\sin\omega t\,dt.$$

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2) Cf. Wang [6], Lemma 7.
3) Cf. Wang [4], Lemma 2.
4) Cf. Matsuyama [2].

Hence we have

$$\int_0^{\pi}\Big[g_{\alpha-1}(t)-g_\alpha(t)\Big]\sin\omega t\,dt=o\Big[(\log\omega)^\alpha/\omega\Big].$$

We put $\gamma(t)=g_{\alpha-1}(t)-g_\alpha(t)$ and

$$\gamma(t)\sim\sum_{n=1}^{\infty}b_n^{(\alpha)}\sin nt,\ t\in(0,\pi),$$

then the above argument shows that $b_n^{(\alpha)}=o[(\log n)^\alpha/n]$. Using the theorem that Fourier series may be integrated term by term, we have

$$\frac{1}{t}\int_0^t\gamma(u)du=\sum_{n=1}^{\infty}b_n^{(\alpha)}\frac{1-\cos nt}{nt}$$

$$=\sum_{nt<1}o[(\log n)^\alpha/n]O(n^2t^2/nt)+\sum_{nt\geq 1}o[(\log n)^\alpha/n]O(1/nt)=o[(\log 1/t)^\alpha].$$

Thus we obtain, by Lemma 3, $\gamma^{1+\delta}(t)=o[t^{1+\delta}(\log 1/t)^\alpha]$.
By Lemma 4, this implies that

$$\delta g_\alpha^{1+\delta}(t)-tg_\alpha^\delta(t)=o[t^{1+\delta}(\log 1/t)^\alpha].$$

On the other hand

$$\delta g_\alpha^{1+\delta}(t)-tg_\alpha^\delta(t)=\delta g_\alpha^{1+\delta}(t)-t\frac{d}{dt}g_\alpha^{1+\delta}(t)=-t^{1+\delta}\frac{d}{dt}\Big[t^{-\delta}g_\alpha^{1+\delta}(t)\Big].$$

Hence $\frac{d}{dt}\Big[t^{-\delta}g_\alpha^{1+\delta}(t)\Big]=o\Big[(\log 1/t)^\alpha\Big]$. But $t^{-\delta}g_\alpha^{1+\delta}(t)=O\Big(\int_0^t\big|g_\alpha(u)\big|du\Big)$ $=o(1)$. Accordingly $t^{-\delta}g_\alpha^{1+\delta}(t)=o\Big\{\int_0^t(\log 1/u)^\alpha du\Big\}=o\Big[t(\log 1/t)^\alpha\Big]$, that is, $g_\alpha^{1+\delta}(t)=o[t^{1+\delta}(\log 1/t)^\alpha]$. Thus, by Lemma 2, we have

$$g_{\alpha+1+\delta'}(t)=o[(\log 1/t)^{\alpha+1+\delta'}],\ \delta'>\delta>0,$$

which is the required result.

7. Further we shall prove the following theorem with stronger assumption and conclusion.

Theorem 4. *If we suppose that*

$$\lim_{t\to 0}\frac{\varphi(t)}{t}=0\qquad (R,\ log,\ \alpha)\tag{7.1}$$

and

$$\lim_{t\to 0}g(t)=0\qquad (R,\ log,\ \alpha),\tag{7.2}$$

then the conjugate derived Fourier series of $f(t)$ is $(R, log, \alpha+1)$ summable to s at the point x, where $1>\alpha>0$ and $g(t)$, $\varphi(t)$ are defined as in § 2.

Proof. Let $\xi(t)=\varphi(t)/t$. Then, by (4.1), we have

$$\frac{\pi}{2}\Big\{R_{\alpha+1}(\omega)-s\Big\}=\frac{\omega}{(\log\omega)^{1+\alpha}}\int_0^{\infty}\xi(t)\Big[(\alpha+1)S_\alpha(\omega t)-S_{\alpha+1}(\omega t)\Big]dt$$

$$=(\alpha+1)I_1-I_2,$$

say. Hence it is sufficient to show that I_1 and I_2 are $o(1)$. For this purpose, we firstly devide I_1 into three parts;

$$I_1=\frac{\omega}{(\log\omega)^{1+\alpha}}\left\{\int_0^{\pi}+\int_{\pi}^{\omega^{\Delta}}+\int_{\omega^{\Delta}}^{\infty}\right\}=I_{1,1}+I_{1,2}+I_{1,3},$$

where $\Delta=(1-\alpha)/\alpha>0$. If we put $\xi^1(t)=\int_0^t\xi(u)du$ and $\xi^*(t)=\int_0^t\left|\xi(u)\right|du$, then

$$I_{1,3}=\frac{\omega}{(\log\omega)^{1+\alpha}}\left[\xi^1(t)S_\alpha(\omega t)\right]_{\omega^{\Delta}}^{\infty}-\frac{\omega^2}{(\log\omega)^{1+\alpha}}\int_{\omega^{\Delta}}^{\infty}\xi^1(t)S_\alpha'(\omega t)\,dt$$

$$=\frac{-\omega}{(\log\omega)^{1+\alpha}}\left[\xi^1(\omega^{\Delta})S_\alpha(\omega^{1+\Delta})\right]+\frac{\omega^2}{(\log\omega)^{1+\alpha}}\int_{\omega^{\Delta}}^{\infty}\frac{1}{(\omega t)^{1+\alpha}}O(1)\,dt,$$

which is $o(1)$ by $\xi^1(t)=O(1)$ and (3.2), and, by (3.1),

$$I_{1,2}=O\left\{\frac{\omega}{(\log\omega)^{1+\alpha}}\int_{\pi}^{\omega^{\Delta}}\left|\xi(t)\right|\frac{(\log\omega t)^{\alpha-1}}{\omega t}dt\right\}$$

$$=O\left\{\frac{1}{(\log\omega)^{1+\alpha}}\left[\xi^*(t)\frac{(\log\omega t)^{\alpha-1}}{t}\right]_{\pi}^{\omega^{\Delta}}\right.$$

$$\left.+\frac{1}{(\log\omega)^{1+\alpha}}\int_{\pi}^{\omega^{\Delta}}\xi^*(t)\left[\frac{(\log\omega t)^{\alpha-1}}{t^2}+\frac{(\log\omega t)^{\alpha-2}}{t^2}\right]dt\right\}=O(1/\log\omega)$$

$$=o(1),$$

since $\xi^*(t)=O(t)$. Thus we have

$$I_1=\frac{\omega}{(\log\omega)^{1+\alpha}}\int_0^{\pi}\xi(t)S_\alpha(\omega t)\,dt$$

$$=o(1)+\Gamma(1+\alpha)\frac{\omega}{(\log\omega)^{1+\alpha}}\int_0^{\pi}\xi_\alpha(u)S_0(\omega u)\,du,$$

similarly as in the proof of (4.3). Thus we have $I_1=o(1)$, as $\omega\to\infty$ (cf. the proof of (4.7)).

On the other hand, using the condition (7.2), we can show similarly as in the proof of Theorem 1 that $I_2=o(1)$. Combining these results, we get the required result.

8. We conclude this paper by stating two theorems of similar type, without proof.

Theorem 5.[5] Let $\varphi(t)=\varphi(x,t)=\frac{1}{2}\left\{f(x+t)+f(x-t)-2s\right\}$. *Suppose that*

$$\int_0^t\varphi_{\alpha-1}(u)\,du=o\left[t(\log 1/t)^{\alpha}\right],\ \textit{as}\ t\to 0,$$

and

$$\int_t^{\pi}\frac{|\varphi_{\alpha-1}(u+t)-\varphi_{\alpha-1}(u)|}{u}du=O\left[(\log 1/t)^{\alpha}\right].$$

Then the necessary and sufficient condition that the Fourier series of

5) Cf. Wang [4], Theorem B and Takahashi [3].

f(t) should be summable (R, log, α), for t=x, to sum s, is that

$$\lim_{t\to 0} \varphi(t)=0 \qquad (R, \log, \alpha),$$

where $\alpha \geqq 1$.

Theorem 6.[6] *Let* $h(t)=(1/\pi)\int_t^\infty \left\{f(x+u)-f(x-u)\right\}\Big/u\,du-s.$

If we suppose that

$$\int_0^t h_\alpha(u)\,du=o\left[t(\log 1/t)^\alpha\right], \text{ as } t\to 0,$$

and

$$\int_t^\pi \frac{|h_\alpha(u+t)-h_\alpha(u)|}{u}\,du=o\left[(\log 1/t)^{\alpha+1}\right], \text{ as } t\to 0,$$

then the conjugate Fourier series of f(t) is (R, log, α+1) summable to sum s at the point x, where $\alpha\geqq 0$.

6) Cf. Wang [7], Theorem 1.