105. On the Property of Lebesgue in Uniform Spaces. III

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In my previous Note (3),[1] we provad a theorem: *If a uniform spxce E is normal and every bounded continuous function is uniformly continuous, then any finite open covering of E has the Lebesgue property.*

First of all, following his valuable advice of Prof. Junji Hashimoto, we shall state the following

Theorem 1. If a uniform space E is normal, and any finite open covering has the Lebesgue property, then every bounded continuous function is uniformly continuous.

The proof is very similar to Theorem 2 of my Note (4), and is contained in it. Therefore it will be omitted.

Throughout the remainder of this Note, we shall give a generalisation of a theorem by A. A. Monteiro and M. M. Peixoto (5), and, as its application, we shall give conditions that uniform spaces be compact (in the sense of N. Bourbaki).

Theorem 2. Let E be a separated uniform space. If any open covering of E has the Lebesgue property and E is precompact,[2] *then E is compact.*

The converse of Theorem 2 is clear from Theorem 3 in my Note (3).

Proof. Let $\mathfrak{F}=\{O_\alpha\}$ be an open covering of E. Since the covering $\mathfrak{F}$ has the Lebesgue property, there is a surrounding V such that $V(x)\subset O_\alpha$, where α depends on x. From the precompactness of E, we can find a finite collection A_i $(i=1,2,\dots,n)$ of E such that $A_i\times A_i\subset V$ $(i=1,2,\dots,n)$ and $\bigcup_{i=1}^n A_i=E$. If $x\in A_i$, then $A_i\subset V(x)$, and hence for each i, there is an index α_i such that $A_i\subset V(x)\subset O_{\alpha_i}$. Therefore, since A_i $(i=1,2,\dots,n)$ is a covering of E, $\bigcup_{i=1}^n O_{\alpha_i}=E$, which prove Theorem 2.

From Theorem 2, we shall have the following

Theorem 3. A necessary and sufficient condition for a separated uniform space E to be compact is that every open covering of E has Lebesgue property and every continuous function of E reaches upper bound.

1) For the undefined terminologies, see my two Notes (3), (4).
2) See the definition in N. Bourbaki (1), Chapter 2.

Proof. It is clear that the condition is necessary. R. Doss (2) proved that if, in a separated uniform space E, every continuous function reaches its upper bound, E is precompact. Therefore the sufficiency of Theorem 3 follows from Theorem 2 and the above-mentioned result of R. Doss.

Before stating Theorem 4, we shall recall the following definition. We call *sequential compact* a separated space in which every filter with countable base has a cluster point, equivalently, every sequence has an accumulation point.

It is well known that sequential compact separable, or metric space, is compact.

Theorem 4. A necessary and sufficient condition for a separated uniform space E to be compact is that any open covering of E has the Lebesgue property and E is sequential compact.

Proof. Let E be a sequential compact uniform space. Then, by P. Samuel's theorem (P. Samuel (6), Theorem XV), E is precompact. (We shall omit the detail of Proof.) Therefore, if any open covering of E has the Lebesgue property, by Theorem 2, E is compact, and the converse of Theorem 4 is trivial. This completes the proof of our theorem.

References

1) N. Bourbaki: Topologie générale, Chap. 1–10, Hermann, Paris (1940–1949).
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5) A. A. Monteiro and M. M. Peixoto: Le nombre de Lebesgue et la continuité uniforme, Port. Math., **10**, 105–113 (1951).
6) P. Samuel: Ultrafilters and compactification of uniform spaces, Trans. Amer. Math. Soc., **64**, 100–132 (1948).