124. On the Property of Lebesgue in Uniform Spaces. IV

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1955)

In this Note, we shall improve the theorem 2 of my first Note [3], and prove that, if any binary covering of a uniform space E has the Lebesgue property, then E is normal.

Theorem 1. The following three properties are equivalent;

(1) *Any binary covering of a uniform space E has Lebesgue property.*

(2) *For every pair of disjoint closed sets F_1, F_2, there is a surrounding V such that $V(F_1)\frown F_2=F_1\frown V(F_2)=0$.*

(3) *For every pair of disjoint closed sets F_1, F_2, there is a surrounding W such that $W(F_1)\frown W(F_2)=0$.*

Proof. We proved the equivalence of (1) and (2) in my first Note [3]. In general, for a surrounding V and a set A of E, there is a surrounding W such that $W(W(A))\subset V(A)$ (for detail, G. Nöbeling [7], p. 169, Axiom U_3).

(2)→(3). Let V be a surrounding mentioned in (2), then we can take a surrounding W such that

$$W(W(F_1))\subset V(F_1).$$

Therefore

$$W(W(F_1))\frown F_2=0.$$

Hence (for detail, G. Nöbeling [7], p. 169, Axiom U_4)

$$W(F_1)\frown W(F_2)=0.$$

(3)→(2). This is trivial.

From the condition (3) of Theorem 1, we have the following.

Theorem 2. If any binary covering of a uniform space E has the property of Lebesgue, E is normal.

Remark. We can prove the equivalence of (1) and (3) by a direct method. The idea of it is in J. Dieudonné [1], p. 72.

By Theorem 2, we can improve Theorem 1 of [5] as follows.

Theorem 3. If every binary covering of E has the Lebesgue property, then any finite covering of E has the property of Lebesgue.

Therefore, Theorem 1 in my Note [3] and Theorem of my Note [5] imply the following.

Theorem 4. Any finite covering of a uniform space E has the Lebesgue property, if and only if E is normal and every bounded continuous function is uniformly continuous.

Recently, S. Kasahara [6] proved the following important theorem.

Theorem 5. If any covering of a uniform space E has the Lebesgue property, then E is paracompact.

J. de Groot and H. de Vries' result [2] and Theorem 5 imply

Theorem 6. If any covering of a locally metrisable space has the Lebesgue property, then E is metrisable (and complete with respect to the metric).

References

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