122. On the Convergence Character of Fourier Series

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1. Let $f(x)$ be an integrable function with period 2π and $s_n(x)$ be the nth partial sum of Fourier series of $f(x)$.

Recently, S. Izumi[1] has proved the following theorem:

If $f(x)$ belongs to the Lip $\alpha(0<\alpha\leqq 1)$ class, then the series[2]

$$\sum_{n=1}^{\infty}|s_n(x)-f(x)|^2/n^{\beta}(\log n)^{\gamma}$$

converges uniformly, where $\beta=1-2\alpha$ and $\gamma>1$ or >2 according as $0<\alpha<1/2$ or $1/2\leqq\alpha<1$.

The object of this paper is to prove the following theorem, which may be partially more general than the above theorem:

Theorem 1. *If $f(x)$ belongs to the Lip $\alpha(0<\alpha<1/2)$ class then the series*

$$\sum_{n=1}^{\infty}\frac{|s_n(x)-f(x)|^k}{n^{\delta}(\log n)^{\gamma}}$$

converges uniformly, where $\delta=1-k\alpha$, $\gamma>1$, $1>k\alpha$, and $k>0$.

Theorem 2.[3] *If $f(x)$ belongs to the Lip α class and if $k\alpha=1$, then the series*

$$\sum_{n=1}^{\infty}\frac{|s_n(x)-f(x)|^k}{(\log n)^{\tau}}$$

converges uniformly, where $\tau>(1-\alpha)/\alpha$ and $k\geqq 2$.

2. For the proof of the theorem we need the following lemma:

Lemma 1. *Under the condition of Theorem 1, we have*

$$\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k=O(n^{1-k\alpha}), \tag{2.1}$$

uniformly.

Lemma 2. *Under the condition of Theorem 2, we have*

$$\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k=O([\log n]^{k-1}),$$

uniformly.

Proof of Lemma 1.[4] We have

$$I=\left(\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k\right)^{1/k}$$

1) S. Izumi: Proc. Japan Acad., **31**, 257–260 (1955).

2) We suppose $1/(\log n)=1$ for $n=1$.

3) This theorem was suggested by Mr. I. Oyama.

4) Cf. A. Zygmund: Trigonometrical series, p. 238, and T. Tsuchikura: Mathematica Japonicae, **1**, 1–5 (1949).

$$(2.2)\qquad =\left\{\sum_{\nu=1}^{n}\left|\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\frac{\sin(\nu+1/2)t}{2\sin t/2}dt\right|^k\right\}^{1/k}$$

$$\leqq\left\{\sum_{\nu=1}^{n}\left|\frac{1}{\pi}\int_0^{1/n}\right|^k\right\}^{1/k}+\left\{\sum_{\nu=1}^{n}\left|\frac{1}{\pi}\int_{1/n}^{\pi}\right|^k\right\}^{1/k}$$

$$=I_1+I_2,$$

say, where $\varphi_x(t)=\varphi(t)=f(x+t)+f(x-t)-2f(x)$. Then

$$I_1^k=O\left\{\sum_{\nu=1}^{n}\nu^k\left(\int_0^{1/n}t^{\alpha}dt\right)^k\right\}=O(n^{1-k\alpha}).$$

For the case $k\geqq 2$, by the Hausdorff-Young inequality we have

$$I_2=O\left\{\left(\int_{1/n}^{\pi}\left|\frac{\varphi(t)}{t}\right|^{k'}dt\right)^{1/k'}\right\},\ (1/k+1/k'=1),$$

$$(2.3)\qquad =O\left\{\left(\int_{1/n}^{\pi}t^{k'(\alpha-1)}dt\right)^{1/k'}\right\},$$

where, by the assumption $1>k\alpha$, $k'(\alpha-1)\neq-1$.
Hence we get

$$I_2=O(n^{1-\alpha-1/k'})=O(n^{1/k-\alpha}).$$

Thus we have Lemma 1 for the case $k\geqq 2$. Let us suppose that $0<e<2$, $k\geqq 2$, then by the Hölder inequality and by the assumption $\alpha<1/2$,

$$\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^e\leqq\left(\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k\right)^{e/k}\cdot n^{1-e/k}$$

$$=O(n^{(1-k\alpha)e/k})\cdot n^{1-e/k}=O(n^{1-e\alpha}).$$

Hence Lemma 1 is also established for the case $0<k<2$.

Proof of Lemma 2. By the above argument, $I_1=O(1)$ and by (2.3)

$$I_2=O\left\{\left(\int_{1/n}^{\pi}t^{k'(\alpha-1)}dt\right)^{1/k'}\right\}=O\left\{\left(\int_{1/n}^{\pi}t^{-1}dt\right)^{1/k'}\right\}$$

$$=O[(\log n)^{1/k'}].$$

Therefore $I^k=O([\log n]^{k/k'})=O([\log n]^{k-1})$, which completes the proof of Lemma 2.

3. **Proof of Theorem 1.** By the Abel transformation, we have

$$\sum_{n=1}^{N}\frac{|s_n(x)-f(x)|^k}{n^\delta(\log n)^\gamma}=\sum_{n=1}^{N-1}\Delta[1/n^\delta(\log n)^\gamma]\cdot\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k$$

$$+\frac{1}{N^\delta(\log N)^\gamma}\sum_{\nu=1}^{N}|s_\nu(x)-f(x)|^k$$

$$=O\left\{\sum_{n=1}^{N-1}n^{1-k\alpha}/[n^{\delta+1}(\log n)^\gamma]\right\}+O\left\{N^{1-k\alpha}/[N^\delta(\log N)^\gamma\right), \text{ by Lemma 1},$$

$$=O\left\{\sum_{n=1}^{N-1}1/[n(\log n)^\gamma]+O\left\{1/(\log N)^\gamma\right\}=O(1).\right.$$

Thus we have Theorem 1.

Proof of Theorem 2. By the same way we have

$$\sum_{n=1}^{N}\frac{|s_n(x)-f(x)|^k}{(\log n)^\tau}=\sum_{n=1}^{N-1}\Delta[1/(\log n)^\tau]\cdot\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k$$
$$+\frac{1}{(\log N)^\tau}\sum_{\nu=1}^{N}|s_\nu(x)-f(x)|^k$$
$$=O\Big\{\sum_{n=1}^{N-1}(\log n)^{k-1}/[n(\log n)^{\tau+1}]\Big\}+O\Big\{(\log N)^{k-1}/(\log N)^\tau\Big\}$$
$$=O\Big\{\sum_{n=1}^{N-1}1/[n(\log n)^{\tau-1/\alpha+2}]\Big\}+O\Big\{1/(\log N)^{\tau-1/\alpha+1}\Big\}$$
$$=O(1),$$

since $\tau-1/\alpha+1>0$, which completes the proof of Theorem 2.

4. Next we shall prove the following

Theorem 3. *If*

$$|f(x+t)-f(x)|=O\Big\{|t|^\alpha\Big/\Big(\log\frac{1}{|t|}\Big)^\gamma\Big\}, \tag{4.1}$$

uniformly, then the series

$$\sum_{n=1}^{\infty}|s_n(x)-f(x)|^k/n^\delta$$

converges uniformly, where $1/2>\alpha>0$, $\delta=1-k\alpha$, $1>k\alpha$, $k>0$, *and* $\gamma>1/k$.

Theorem 4. *If $f(x)$ satisfies (4.1) and if $k\alpha=1$, then the series*

$$\sum_{n=1}^{\infty}|s_n(x)-f(x)|^k$$

converges uniformly, where $k\geqq 2$ *and* $1+k(\gamma-1)>0$.

The proof of Theorem 3 may be done by the following lemma, as in the proof of Theorem 1.

Lemma 3. *Under the assumption of Theorem 3, we have*

$$\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k=O[n^{1-k\alpha}/(\log n)^{k\gamma}]. \tag{4.2}$$

Proof of Lemma 3. If (4.2) is established for $k\geqq 2$, then it holds a fortiori for every $0<k<2$. Hence we may suppose $k\geqq 2$ (cf. Proof of Lemma 1). We divide I, which is denoted by (2.2), into following three parts;

$$I\leqq\Big\{\sum_{\nu=1}^{n}\Big|\frac{1}{\pi}\int_0^{1/n}\Big|^k\Big\}^{1/k}+\Big\{\sum_{\nu=1}^{n}\Big|\frac{1}{\pi}\int_{1/n}^{1/n^\mu}\Big|^k\Big\}^{1/k}+\Big\{\sum_{\nu=1}^{n}\Big|\frac{1}{\pi}\int_{1/n^\mu}^{\pi}\Big|^k\Big\}^{1/k}$$
$$=I_1+I_2'+I_3,$$

where $0<\mu<\min.\{1,(1/k-\alpha)\}$. By the assumption, we have

$$I_1^k=O\Big\{\sum_{\nu=1}^{n}\nu^k\Big[\int_0^{1/n}t^\alpha\Big/\Big(\log\frac{1}{t}\Big)^\gamma dt\Big]^k\Big\}=O[n^{1-k\alpha}/(\log n)^{k\gamma}]$$

and, by the Hausdorff-Young inequality,

$$I_2'=O\Big\{\int_{1/n}^{1/n^\mu}\Big|\frac{\varphi(t)}{t}\Big|^{k'}dt\Big\}^{1/k'}=O\Big\{\int_{1/n}^{1/n^\mu}t^{(\alpha-1)k'}\Big/\Big(\log\frac{1}{t}\Big)^{k'\gamma}dt\Big\}^{1/k'}$$
$$=O\Big\{\frac{1}{(\log n)^\gamma}\Big[\int_{1/n}^{1/n^\mu}t^{(\alpha-1)k'}dt\Big]^{1/k'}\Big\}$$

$$=O\left\{\frac{1}{(\log n)^{\gamma}}n^{1-\alpha-1/k'}\right\}=O\left\{n^{1/k-\alpha}/(\log n)^{\gamma}\right\}.$$

By the same way,

$$I_3=O\left\{\int_{1/n^{\mu}}^{\pi}\left|\frac{\varphi(t)}{t}\right|^{k'}dt\right\}^{1/k'}=O(n^{\mu}).$$

Summing up above estimations, we get the required.

Now we prove Theorem 4. By the assumption, we easily see that

$$I_1^k=O[1/(\log n)^{k\gamma}]$$

and

$$I_2=O\left\{\int_{1/n}^{\pi}t^{k'(\alpha-1)}\Big/\left(\log\frac{1}{t}\right)^{k'\gamma}dt\right\}^{1/k'}$$

$$=O\left\{\int_{1/n}^{\pi}\frac{dt}{t(\log 1/t)^{k'\gamma}}\right\}^{1/k'}=O(1),$$

for $k'\gamma=\gamma k/(k-1)>1$.

Hence we get the theorem.

5. Our theorems stated above may be extended. For example, we have

Theorem 5. *If*

$$\int_0^t|\varphi_x(u)/u^{\alpha}|^2\,du=O(t),\qquad \textit{uniformly},$$

then the series

$$\sum_{n=1}^{\infty}\frac{|s_n(x)-f(x)|^2}{n^{\beta}(\log n)^{\gamma}}$$

converges uniformly, where $\beta=1-2\alpha>0$ *and* $\gamma>1$.

Theorem 6. *If* $f(x)$ *belongs to the* $Lip(\alpha,p)$ *class, then the series*

$$\sum_{n=1}^{\infty}\frac{|s_n(x)-f(x)|^k}{n^{\delta}(\log n)^{\gamma}}$$

converges almost everywhere, where $k>0$, $p>1$, $p>k$, $1>\alpha>0$, $\delta=1-k\alpha$, *and* $\gamma>1$.

Theorem 7. *If*

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p\,dx\right)^{1/p}=O\left[|t|^{\alpha}\Big/\left(\log\frac{1}{|t|}\right)^{\gamma}\right],$$

then the series

$$\sum_{n=1}^{\infty}\frac{|s_n(x)-f(x)|^k}{n^{\delta}}$$

converges almost everywhere, where $k>0$, $p>1$, $p>k$, $1>\alpha>0$, $\gamma>1/k$, *and* $\delta=1-k\alpha$.

The proof of Theorem 5 is similar to that of Theorem 1,[5] and the proof of Theorems 6 and 7 runs similarly as in the theorem of S. Izumi.[6] Hence we omitt the detail.

5) Cf. G. Alexits: Acta Szeged, **3**, 32–37 (1927).
6) S. Izumi: To appear.