145. Note on Commutator Subgroups of Factorisable Groups

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1. A group G is called factorisable by its subgroups A and B if each element g of G may be written in the form $g=ab$ with a in A, b in B (written $G=AB$). Since G. Zappa [1] considered factorisable groups, many interesting results have been obtained by several authors. It seems to the author, however, that the structure of commutator subgroups of factorisable groups is not well known. In this note we shall study the structure of derived groups of factorisable groups by certain subgroups. Let (A, B) denote the subgroup generated by all the elements of the form $aba^{-1}b^{-1}$, where a and b run all the elements of A and B respectively. Our main result which plays an important rôle in the sequel study is stated as follows: *if a group G is factorisable by two subgroups A and B, then the subgroup (A, B) is always normal in G.*

2. **Theorem 1.** *Let G be a factorisable group by two arbitrary subgroups A and B, i.e. $G=AB$. Then the subgroup (A, B) is normal in G.*

Proof. Let α be an element of A and let $aba^{-1}b^{-1}$ be an element of $(A, B)=(B, A)$, where $a \in A$ and $b \in B$. Then we have

$$\alpha^{-1}aba^{-1}b^{-1}\alpha=\alpha^{-1}aba^{-1}\alpha b^{-1}b\alpha^{-1}b^{-1}\alpha$$
$$=(\alpha^{-1}a)b(\alpha^{-1}a)^{-1}b^{-1}\cdot b\alpha^{-1}b^{-1}\alpha.$$

Therefore, this element belongs to (A, B). Similarly, for any element β of B, we obtain that

$$\beta^{-1}aba^{-1}b^{-1}\beta=\beta^{-1}a\beta a^{-1}\cdot a(\beta^{-1}b)a^{-1}(\beta^{-1}b)^{-1};$$

hence $\beta^{-1}aba^{-1}b^{-1}\beta \in (A, B)$, q.e.d.

Lemma. *Let G be a factorisable group by an abelian subgroup A and an arbitrary subgroup B, and let G' and B' be the commutator subgroups of G and B respectively. Then*

$$G'=B'(A, B).$$

Proof. It is obvious that $G'=(AB, AB) \supseteq B'(A, B)$. We shall prove $G' \subseteq B'(A, B)$. Let g and h be two elements of G such that $g=ab$, $h=a'b'$, $a, a' \in A$, $b, b' \in B$. Since there exist α and β such that $ba'=\alpha\beta$, $\alpha \in A$, $\beta \in B$, we have

$$\begin{aligned}(g, h)&=ghg^{-1}h^{-1}\\ &=aba'b'b^{-1}a^{-1}b'^{-1}a'^{-1}\\ &=(a\alpha)(\beta b'b^{-1})a^{-1}b'^{-1}a'^{-1}\\ &\equiv(\beta b'b^{-1})(a\alpha)a^{-1}b'^{-1}a'^{-1} \qquad (\text{mod. } (A, B))\\ &=\beta b'b^{-1}\alpha b'^{-1}a'^{-1}\end{aligned}$$

$$=\beta b'b^{-1}\cdot ba'\beta^{-1}\cdot b'^{-1}a'^{-1}$$
$$=\beta b'\beta^{-1}b'^{-1}\cdot(b'\beta)a'(b'\beta)^{-1}a'^{-1}.$$

Thus, we obtain $G'\subseteq B'(A, B)$, q.e.d.

If A and B are both abelian, then $G'=(A, B)$. Hence we obtain the following theorem by a simple calculation, which was reported by N. Itô [2].

Theorem 2. *Any factorisable group by two abelian subgroups is always soluble of rank* 2.

Theorem 3. *Let G be a factorisable group by its abelian subgroup A and its soluble subgroup B of rank 2. If G is soluble, then*

$$rank(G)\leq rank((A, B))+2,$$

where rank(G) means the length of derived series of G.

Proof. Since B' is abelian, we can apply above Lemma for $G'=B'(A, B)$. Therefore, we obtain

$$G''=(B', (A, B))(A, B)'.$$

Since (A, B) is normal in G by Theorem 1, G'' is contained in (A, B). On the other hand, by our assumption (A, B) is also soluble. Hence, there exists an integer m such that $(A, B)^{(m)}=1$. These imply that $(G'')^{(m)}=G^{(m+2)}=1$. This completes the proof.

Theorem 4. *If a factorisable group $G=AB$ satisfies the following conditions:*

(1) *A is an abelian subgroup,*

(2) *B is a nilpotent subgroup with class 2,*

(3) *$AB'=B'A$, i.e. AB' forms a subgroup in G,*

then

$$G^{(i)}=(B', N, N', \dots, N^{(i-2)})N^{(i-1)}$$

with the abelian normal subgroup $(B', N, N', \dots, N^{(i-2)})$ in $G^{(i)}$, where $G^{(i)}$ and N denote the i-th derived group of G and (A, B) respectively.

Proof. We shall prove by induction on the number i of the i-th derived group $G^{(i)}$.

First we shall prove the theorem for $i=2$. We can easily obtain $G''=(B', N)N'$ by above Lemma. Then, we shall prove that (B', N) is abelian. For any element (α_1, b_1) of $(B', N)=(N, B')$, where $\alpha_1=(a, b)\in(A, B)=N$ and $b_1\in B'$, we obtain that

$$(\alpha_1, b_1)=\alpha_1 b_1\alpha_1^{-1}b_1^{-1}$$
$$=aba^{-1}b^{-1}\cdot b_1\cdot bab^{-1}a^{-1}\cdot b_1^{-1}$$
$$=(aba^{-1})b_1(aba^{-1})^{-1}b_1^{-1}.$$

Since there exist α and β in A and B respectively such that $aba^{-1}=\alpha\beta$, we have

$$(\alpha_1, b_1)=\alpha\beta b_1\beta^{-1}\alpha^{-1}b_1^{-1}$$
$$=\alpha b_1\alpha^{-1}b_1^{-1}.$$

Therefore, (α_1, b_1) lies in (A, B'). This means $(B', N)\subseteq(A, B')$. On the other hand, by the condition (3) and Theorem 2, (A, B') is abelian. Hence, (B', N) is abelian. Moreover, (B', N) is normal in G'' since it is normal in G' by Theorem 1.

Now, assume that $G^{(i)}=(B', N, N', \dots, N^{(i-2)})N^{(i-1)}$ and that $(B', N, N', \dots, N^{(i-2)})$ is abelian and normal in $G^{(i)}$. We put $(B', N, N', \dots, N^{(i-2)})=K$. By above Lemma we have immediately

$$G^{(i+1)}=(K, N^{(i-1)})N^{(i)}.$$

By induction hypothesis, K is invariant under the transformation of any element of $N^{(i-1)}$. Then we have

$$(K, N^{(i-1)}) \subseteq K.$$

Hence, $(K, N^{(i-1)})$ is abelian. Moreover, $(K, N^{(i-1)})$ is normal in $G^{(i+1)}$ since it is normal in $G^{(i)}$ by Theorem 1.

Thus the proof of Theorem 4 is completed.

3. By Theorem 2, if A and B are both abelian, then

$$\begin{aligned}\operatorname{rank}(G) \leq 2 = 1+1 &= \operatorname{rank}((A, B)) + \operatorname{rank}(B) \\ &= \operatorname{rank}((A, B)) + \operatorname{class}(B).\end{aligned}$$

Moreover, in Theorem 4, since G is soluble by N. Itô [3], N is also soluble. Hence there exists an integer m such that $N^{(m-2)}=1$. This implies $G^{(m)}=1$. Thus we have

$$\operatorname{rank}(G) \leq \operatorname{rank}((A, B)) + 2 = \operatorname{rank}((A, B)) + \operatorname{class}(B).$$

When $\operatorname{rank}(B)$ or $\operatorname{class}(B)$ is not greater than 2, it follows from these facts that

$$\begin{aligned}\operatorname{rank}(G) &\leq \operatorname{rank}((A, B)) + \operatorname{rank}(B) \\ \text{or } &\leq \operatorname{rank}((A, B)) + \operatorname{class}(B).\end{aligned}$$

But, in general, when the $\operatorname{rank}(B)$ or $\operatorname{class}(B)$ is greater than 2, it seems very difficult to prove this inequality unless the relations between the elements of A and B are concretely given.

References

[1] G. Zappa: Costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro, Atti Secondo Congresso Unione Mat. Italiana Bologna, 115–125 (1940).

[2] N. Itô: Über das Produkt von zwei abelschen Gruppen, Math. Z., **63**, 400–401 (1955).

[3] N. Itô: Remarks on factorisable groups, Acta Uni. Szeged, **14**, 83–84 (1951).