4. On Images of an Open Interval under Closed Continuous Mappings

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1. Introduction. A mapping of a topological space X onto another topological space Y is said to be closed if the image of every closed subset of X is closed in Y. As is well known, in order that a metric space Y be the image of the closed line interval $[0,1]$ under a closed continuous mapping it is necessary and sufficient that Y be a locally connected continuum.

In the present note we shall establish the following theorem, which is an analogue of the celebrated theorem of Hahn-Mazurkiewicz mentioned above and may be considered as a generalization of it since any closed continuous mapping of the open line interval $(0,1)$ onto a locally connected continuum can be extended over $[0,1]$ by our Theorem 3 below.

Theorem 1. *In order that a metric space Y be the image of the open line interval* $(0,1)$ *under a closed continuous mapping it is necessary and sufficient that Y be a separable, locally compact, connected, locally connected space with at most two end-points in the sense of Freudenthal (i.e. $\gamma(Y)-Y$ consists of at most two points).*

Here $\gamma(Y)$ means the Freudenthal compactification of the space Y (cf. [1], [2]).

For any positive integer m let Q_m be the union of m closed segments $a_i a_0$, $i=1, 2, \cdots, m$, each two having only one point a_0 in common, and let P_m be the space obtained from Q_m by removing the points a_i, $i=1, 2, \cdots, m$. Then P_1 and P_2 are homeomorphic to $(0,1]$ and $(0,1)$ respectively, and hence Theorem 1 is contained in the following theorem.

Theorem 2. *In order that a metric space Y be the image of P_m under a closed continuous mapping it is necessary and sufficient that Y be a separable, locally compact, connected, locally connected space with at most m end-points in the sense of Freudenthal.*

2. Lemmas. We shall first prove

Lemma 1. *Let f be a closed continuous mapping of a metric space X onto another metric space Y. If A is a closed subset of Y whose boundary $\mathfrak{B}A$ is compact, then $\mathfrak{B}f^{-1}(A)$ is also compact.*

Proof. Let us put

$$V_i = A \cup \{y \mid \rho(y, \mathfrak{B}A) < 1/i\},$$

where ρ denotes a metric of Y (in case $\mathfrak{B}A=0$ we put $V_i=A$). Then V_i is open and $\{V_i \mid i=1, 2, \cdots\}$ has the property that for any open set H containing A there exists some V_i with $V_i \subset H$; this is seen from the compactness of $\mathfrak{B}A$.

Suppose that $\mathfrak{B}f^{-1}(A)$ is not compact. Then there exist a countable number of points $x_i, i=1, 2, \cdots$, of $\mathfrak{B}f^{-1}(A)$ such that $\{x_i\}$ has no limit point. Then we can find a discrete collection $\{G_n\}$ of open sets of X such that

$$x_i \in G_i \quad \text{for } i=1, 2, \cdots; \; G_i \frown G_j=0 \quad \text{for } i \neq j$$

and $\{G_n\}$ is locally finite. Since each point x_i belongs to the boundary of $f^{-1}(A)$, there exists a point x_i' of X such that

$$x_i' \notin f^{-1}(A), \quad x_i' \in G_i \frown f^{-1}(V_i).$$

Then the set C consisting of all points x_i', $i=1, 2, \cdots$ is closed, and hence if we put $H=Y-f(C)$, H is an open set of Y. Since $x_i' \notin f^{-1}(A)$, we have $A \subset H$. Hence there exists some V_i such that $V_i \subset H$. This implies that $f(x_i') \notin V_i$ for some i. On the other hand we have chosen the point x so that $x_i' \in f^{-1}(V_i)$. This is a contradiction. Thus Lemma 1 is proved.

Lemma 2. *Let f be a closed continuous mapping of a metric space X onto another metric space Y. If X is separable or locally compact or locally connected, so also is Y.*

This is proved for the first two properties by S. Hanai and the author [3], and for the last property by R. L. Wilder [5] and G. T. Whyburn [4].

Lemma 3. *Let R be a metric space which is a locally connected continuum. Let p be not a locally separating point of R*[1] *and let q be any point distinct from p. Then for any positive number ε there exists a finite ε-covering $\{K_1, \cdots, K_m\}$ of R such that each K_i is a locally connected continuum and any two consecutive sets K_i, K_{i+1} have at least one common point and*

$$p \in K_1, \; p \notin K_i \quad \text{for } i \geqq 2; \; q \in K_m.$$

Proof. By [5, Theorems III, 3.4, 3.6] there exists an ε-covering $\{L_1, \cdots, L_s\}$ of R such that each L_i is a locally connected continuum and L_1 is the only set of the covering which contains p. Then there exists an open connected set V_0 such that $\overline{V}_0$ is locally connected and $p \in V_0$, $\overline{V}_0 \frown L_i=0$ for $i>1$, by the same theorems quoted above; likewise there exists an open connected set G of diameter $<\varepsilon$ such that $L_1 \subset G$. Since $G-p$ is connected and L_1-V_0 is compact, there exist a finite number of locally connected continua $C_1, \cdots, C_k$ such

1) That is, $G-p$ is connected for any open connected set G of R; for this it is sufficient that there exists a basis $\{W_\alpha\}$ for neighborhoods of p such that $W_\alpha - p$ is connected for each α.

that $L_1 - V_0 \subset \bigcup_{i=1}^{k} C_i \subset G-p$. Since $G-p$ is arcwise connected these continua are joined by arcs in $G-p$ which, together with $C_1, \cdots, C_k$, form a locally connected continuum K_2 (cf. [5, Theorem III, 3.15]). We arrange the sets $K_2, L_2, L_3, \cdots, L_s$ (with repetitions) as a chain $\{K_2, K_3, \cdots, K_m\}$ (K_i being some L_j or K_2) which begins with K_2 and ends with K_m containing q. If we put $K_1 = \overline{V}_0$, then $\{K_1, K_2, \cdots, K_m\}$ has the desired properties.

Lemma 4. *Under the same assumptions of Lemma 3, there exists a continuous mapping f of the closed interval $[0,1]$ onto R such that $f(0)=p$, $f(1)=q$ and $f(t) \neq p$ for $t>0$; the partial mapping $f_0 = f \mid (0,1]$ is a closed continuous mapping of the semi-open interval $(0,1]$ onto $R-p$.*

Proof. Applying Lemma 3 repeatedly we can find a countable number of locally connected subcontinua

$$K(i_1, \cdots, i_m), \quad i_1 = 1, \cdots, s_1; \quad i_k = 1, \cdots, s(i_1, \cdots, i_{k-1}); \quad k=2, \cdots, m; \quad m=1, 2, \cdots$$

of R, where repetitions of the same set are allowed and $s_1 \geqq 2$, $s(i_1, \cdots, i_{n-1}) \geqq 2$, with the following properties:

(1) $\mathfrak{K}_m = \{K(i_1, \cdots, i_m) \mid i_1, \cdots, i_m\}$ is a 2^{-m}-covering of R for each m.

(2) Let us define an order among the elements of $\mathfrak{K}_m$ as follows: $K(i_1 \cdots, i_m) < K(j_1, \cdots, j_m)$ if $i_1 < j_1$, or $i_r = j_r$ for $r=1, \cdots, n-1$ and $i_n < j_n$ for some n with $n \leqq m$; then any two consecutive sets in this order have a point in common and the first element $K(1, \cdots, 1)$ is the only set of $\mathfrak{K}_m$ containing p and the last element $K(s_1, s(s_1), \cdots, s(s_1, s(s_1), \cdots))$ contains q.

(3) $K(i_1, \cdots, i_m) = \bigcup \{K(i_1, \cdots, i_m, i_{m+1}) \mid i_{m+1} = 1, \cdots, s(i_1, \cdots, i_m)\}$.

Corresponding to this series of subdivisions of R we can construct a countable number of closed subintervals

$$T(i_1, \cdots, i_m), \quad i_1 = 1, \cdots, s_1; \quad i_k = 1, \cdots, s(i_1, \cdots, i_{k-1}); \; k=2, \cdots, m; \quad m=1, 2, \cdots$$

of $[0, 1]$ such that these intervals satisfy the conditions (1) to (3), with $K(i_1, \cdots, i_m)$ replaced by $T(i_1, \cdots, i_m)$ and with p, q replaced by 0,1 in $[0,1]$ respectively, and an additional condition that two sets $T(i_1, \cdots, i_m)$, $T(j_1, \cdots, j_m)$ have only one common point or no common point according as they are consecutive sets or not.

For any real number t such that $0 \leqq t \leqq 1$, there exists a system $(i_1, i_2, \cdots)$ such that $t \in T(i_1, \cdots, i_m)$ for $m=1, 2, \cdots$. Then $\bigcap \{K(i_1, i_2, \cdots, i_m) \mid m=1, 2, \cdots\}$ consists of a single point which we will denote by $f(t)$. The mapping f is easily seen to be a single-valued continuous mapping from $[0,1]$ onto R satisfying the condition of Lemma 4.

3. The Freudenthal compactification. We recall the definition of the Freudenthal compactification [1] by our method given in a previous paper [2].

Let R be a semicompact Hausdorff space (i.e. every point of R has arbitrarily small neighborhoods with compact boundaries). Let $\mathfrak{M}$ be the totality of all finite open coverings $\{G_1,\cdots,G_s\}$ of R such that $\mathfrak{B}G_i$ are compact. Then R is completely regular and $\mathfrak{M}$ is a completely regular uniformity of R agreeing with the topology of R. Let S be the completion of R with respect to the uniformity $\mathfrak{M}$. Then S has the following properties:

(a) S is a compact Hausdorff space containing R as a dense subset.

(b) For any point p of S and for any neighborhood U of p there exists an open set V containing p such that $V\subset U$ and $\mathfrak{B}V\subset R$.

(c) For any two open sets G and H of R with compact boundaries, $(G\smile H)^*=G^*\smile H^*$ where we put $A^*=S-R-A$ for any open set A of R.

Conversely, the properties (a), (b) and (c) characterize S. This space S is denoted by $\gamma(R)$; we call $\gamma(R)$ the Freudenthal compactification of R. Each point of $\gamma(R)-R$ is called an end-point of R in the sense of Freudenthal.

Lemma 5. *If G is an open connected subset of $\gamma(R)$ such that $\mathfrak{B}G\subset R$, then $G\frown R$ is an open connected subset of R.*

Proof. Suppose that there exist two open subsets H_1, H_2 of R such that $G\frown R=H_1\smile H_2$, $H_1\frown H_2=0$. Then the boundary of $G\frown R$ in the space R is compact and hence the boundary of H_i in the space R is likewise compact since $\overline{H}_i\frown R-H_i=\overline{H}_i\frown R-(H_1\smile H_2)\subset(\overline{G}-G)\frown R$. Therefore we have from (c) $G\subset(G\frown R)^*=(H_1\smile H_2)^*=H_1^*\smile H_2^*$, $H_1^*\frown H_2^*=(H_1\frown H_2)^*=0$. Since by the assumption G is connected we have $G\frown H_i^*=0$ for some i and hence $H_i=G\frown H_i=(G\frown H_i^*)\frown R=0$. This proves our lemma.

Theorem 3. *Let f be a closed continuous mapping of a semicompact metric space X onto a semicompact metric space Y. Then f can be extended to a continuous mapping of $\gamma(X)$ onto $\gamma(Y)$; in particular the number of end-points of X is not less than that of end-points of Y.*

Proof. Let $\{H_1,\cdots,H_m\}$ be any finite open covering of Y such that $\mathfrak{B}H_i$ is compact for each i. Then there exists an open covering $\{K_1\cdots,K_m\}$ such that $\overline{K}_i\subset H_i$ and $\mathfrak{B}K_i$ is compact for each i (cf. [2, Lemma 1]). Since $\mathfrak{B}\overline{K}_i\subset\mathfrak{B}K_i$, by Lemma 1 we see that $\mathfrak{B}f^{-1}(\overline{K}_i)$ is compact. Therefore if we put $G_i=\operatorname{Int} f^{-1}(\overline{K}_i)$, $i=1,2,\cdots,m$, $\{G_1,\cdots,G_m\}$ is a finite open covering of X such that $f^{-1}(K_i)\subset G_i\subset f^{-1}(H_i)$ and $\mathfrak{B}G_i$ is compact for each i.

If we consider X and Y uniform spaces with the uniformities consisting of all finite open coverings by open sets with compact boundaries, the above consideration shows that f is a uniformly continuous mapping. Hence f can be extended to a continuous mapping of $\gamma(X)$ onto $\gamma(Y)$.

4. Proof of Theorem 2. The necessity assertion of Theorem 2 readily follows from Lemma 2 and Theorem 3 since the number of end-points of P_m is m.

Let Y be a separable, locally compact, connected, locally connected metric space (i.e. a locally connected generalized continuum) with m end-points in the sense of Freudenthal. Then $\gamma(Y)$ is a connected, compact metrizable space (cf. [1], [2]) and hence we shall treat $\gamma(Y)$ as a metric space. Furthermore $\gamma(Y)$ is locally connected since dim $(\gamma(Y)-Y)\leqq 0$ (cf. [5]).

Since each end-point is not a locally separating point of $\gamma(Y)$ by Lemma 5, we can prove similarly as in the proof of Lemma 3 that there exists a finite covering $\{K_1, K_1', K_2, K_2', \cdots, K_m, K_m', L_1, \cdots, L_n\}$ of $\gamma(Y)$ by locally connected continua, such that each K_i contains exactly one end-point, any set of the covering other than $K_i (i=1,\cdots,m)$ contains no end-point, and each K_i does not intersect any set of the covering other than K_i' $(i=1,\cdots,m)$. If we denote by K_0 the union of $K_1',\cdots,K_m'$, $L_1,\cdots,L_n$, we have the covering $\{K_0, K_1,\cdots,K_m\}$ of $\gamma(Y)$ by locally connected continua such that each K_i for $i\geqq 1$ contains exactly one end-point and K_0 contains no end-point, and $K_0\frown K_i\neq 0$ for $i\geqq 1$.

Let P_m and Q_m have the same meaning as in the introduction. We take a point y_0 in K_0. Then for each i there exists, by Lemma 4, a closed continuous mapping g_i of $a_ia_0-a_i$ onto $(K_i\smile K_0)\frown Y$ such that $g_i(a_0)=y_0$. Let f be a mapping of P_m onto Y such that for each i the partial mapping $f\mid a_ia_0-a_i$ coincides with g_i. Then f is clearly a closed continuous mapping of P_m onto Y.

For a positive integer n with $n<m$ there exists obviously a closed continuous mapping of P_m onto P_n and likewise a closed continuous mapping of P_m onto the closed line interval [0,1]. Thus the sufficiency assertion of Theorem 3 is completely proved.

References

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