22. *Some Trigonometrical Series. XX*

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1. S. Chowla [1] proposed the problem (the cosine problem):

Let K be an arbitrary positive number. Then can we find $N=N(K)$ such that

$$\min_{0\leqq x<2\pi} (\cos m_1x+\cos m_2x+\cdots+\cos m_nx)<-K \tag{1}$$

for all $n\geqq N$ where $(m_1, m_2,\cdots, m_n)$ is any set of n different positive integers?

We shall give an answer to this problem in a special case and prove theorems closely connected.

2. Theorem 1. *For any positive number K, there is an N such that (1) holds for all $n\geqq N$ and for any set of distinct integers $(m_1, m_2,\cdots, m_n)$ such that the number of solutions of the equations*

$$m_i+m_j=m_k \qquad (i<j<k\leqq n) \tag{2}$$

is of $o(n^2)$.

Proof. If not so, there is a positive number K such that for any N there are an $n\geqq N$ and a set of distinct integers $(m_1, m_2, \cdots, m_n)$ such that

$$-f(x)=\sum_{k=1}^{n}\cos m_kx\geqq -K.$$

Then we have

$$-n\leqq f(x)\leqq K.$$

Let us consider the integral

$$I=\int_0^{2\pi}(K-f)^2(n+f)dx$$

which was used by S. Szidon [2] for a different purpose. By the Schwarz inequality

$$I^2=\left(\int_0^{2\pi}(K-f)^{3/2}\cdot(K-f)^{1/2}(n+f)dx\right)^2 \tag{3}$$

$$\leqq\int_0^{2\pi}(K-f)^3dx\int_0^{2\pi}(K-f)(f+n)^2dx=I_1\cdot I_2,$$

say. Now since

$$\int_0^{2\pi}f\,dx=0,\quad \int_0^{2\pi}f^2dx=\pi n^2,$$

we have

$$I=\int_0^{2\pi}(K^2-2Kf+f^2)(n+f)dx$$

$$=2\pi K^2 n+n\int_0^{2\pi} f^2 dx-2K\int_0^{2\pi} f^2 dx+\int_0^{2\pi} f^3 dx$$
$$=(\pi n^2+2\pi K^2 n-2\pi Kn)+\int_0^{2\pi} f^3 dx=\alpha_n+\int_0^{2\pi} f^3 dx$$

and then

$$I^2=\alpha_n^2+2\alpha_n\int_0^{2\pi} f^3 dx+\Big(\int_0^{2\pi} f^3 dx\Big)^2.$$

We have also

$$I_1=\int_0^{2\pi}(K^3-3K^2 f+3Kf^2-f^3)dx$$
$$=(3\pi Kn+2\pi K^3)-\int_0^{2\pi} f^3 dx=\beta_n-\int_0^{2\pi} f^3 dx$$

and

$$I_2=\int_0^{2\pi}(K-f)(n^2+2nf+f^2)dx$$
$$=(2\pi Kn^2-2\pi n^2+\pi Kn)-\int_0^{2\pi} f^3 dx=\gamma_n-\int_0^{2\pi} f^3 dx.$$

By (3) we obtain

$$(2\alpha_n+\beta_n+\gamma_n)\int_0^{2\pi} f^3 dx\leqq \beta_n\gamma_n-\alpha_n^2$$

where

$$2\alpha_n+\beta_n+\gamma_n=2\pi Kn^2+4\pi K^2 n+2\pi K^3$$
$$=2\pi K(n^2+2Kn+K^2)=2\pi K(n+K)^2,$$
$$\beta_n\gamma_n-\alpha_n^2=\pi^2 n[-n^3+2K(K-1)n^2+4K^2(K-1)n+2K^4].$$

Hence, for large n,

$$(4)\qquad -\int_0^{2\pi} f^3 dx\geqq \pi(1+o(1))n^2/2K.$$

On the other hand

$$-\int_0^{2\pi} f^3 dx=\int_0^{2\pi}\Big(\sum_{i=1}^{n}\cos^2 m_i x+2\sum_{\substack{i,j=1\\ i<j}}^{n}\cos m_i x\cos m_j x\Big)\sum_{k=1}^{n}\cos m_k x\,dx$$
$$=\int_0^{2\pi}\Big\{\sum_{\substack{i,j=1\\ i<j}}^{n}(\cos(m_i-m_j)x+\cos(m_i+m_j)x)$$
$$-\frac{1}{2}\sum_{i=1}^{n}\cos 2m_i x\Big\}\sum_{k=1}^{n}\cos m_k x\,dx.$$

Let $S=S(n)$ be the number of solutions of (2), then

$$-\int_0^{2\pi} f^3 dx\leqq 2\pi S+\pi n/2,$$

which contradicts (4), since $S=o(n^2)$.

3. Theorem 2. *Let* $0<\alpha<1/2$. *If*

$$\sum_{k=1}^{n}\cos m_k x\geqq -n^\alpha,$$

then the number of solutions of (2) is greater than

$$\frac{1}{4}n^{2-\alpha}(1+o(1))$$

as $n\to\infty$.

Proof. Let $K=n^{\alpha}$ in the proof of Theorem 1, then

$$2\alpha_n+\beta_n+\gamma_n=2\pi n^{2+\alpha}(1+o(1)),$$
$$\beta_n\gamma_n-\alpha_n^2=-\pi^2n^4(1+o(1))$$

and then

$$-\int_0^{2\pi}f^3dx\geqq(\pi/2)n^{2-\alpha}(1+o(1)).$$

Thus we get the theorem as in § 2.

4. Theorem 3. *For any positive number K, there is an N such that (1) holds for all* $n\geqq N$ *and for any set of distinct integers* $(m_1, m_2,\cdots, m_n)$ *such that the number of solutions of the equations*

(5) $$m_i\pm m_j=m_k\pm m_l \qquad (i<j\leqq n,\ k<l\leqq n)$$

is of $o(n^3)$.

Proof. As in the proof of Theorem 1, if not so, there is a positive number K such that for any N there are an $n\geqq N$ and a set of distinct integers $(m_1, m_2,\cdots, m_n)$ such that

$$-f(x)=\sum_{k=1}^{n}\cos m_kx\geqq -K,$$

and then

$$-n\leqq f(x)\leqq K.$$

Let us consider the integral

$$J=\int_0^{2\pi}(K-f)^3(n+f)dx$$

whose square is less than

$$\int_0^{2\pi}(K-f)^4dx\int_0^{2\pi}(K-f)^2(n+f)^2dx=J_1\cdot J_2,$$

say. Now

$$J=(3\pi Kn^2+2\pi K^3n-3\pi K^2n)-(n-3K)\int_0^{2\pi}f^3dx-\int_0^{2\pi}f^4dx,$$

$$J_1=(2\pi K^4+6\pi K^2n)-4K\int_0^{2\pi}f^3dx+\int_0^{2\pi}f^4dx,$$

$$J_2=(\pi n^3+2\pi K(K-2)n^2+\pi K^2n)+(2n-2K)\int_0^{2\pi}f^3dx+\int_0^{2\pi}f^4dx.$$

Hence the inequality $J^2\leqq J_1\cdot J_2$ becomes

$$3\pi^2K^2n^4-2\pi^2K^3(K-3)n^3-\pi^2K^4(4K-3)n^2-2\pi^2Kn$$
$$+(-2\pi Kn^3+4\pi K^2(K-1)n^2+2\pi K^3(4K+1)n+4\pi K^5)\int_0^{2\pi}f^3dx$$
$$+(n^2+2Kn+K^2)\left(\int_0^{2\pi}f^3dx\right)^2$$

$$\leqq \pi(n^3+2K^2n^2-2Kn^2+K^2n+4K^3n+2K^4)\int_0^{2\pi} f^4 dx.$$

By the inequality (4)

$$\int_0^{2\pi} f^4 dx \geqq \frac{\pi}{4K^2} n^3(1+o(1))$$

which contradicts the assumption.

Theorem 4. *Let* $0<\alpha<1/2$. *If*

$$\sum_{k=1}^{n} \cos m_k x \geqq -n^\alpha,$$

then the number of solutions of (5) is greater than

$$\frac{1}{4} n^{3-2\alpha}(1+o(1)).$$

Proof is similar to the case of Theorem 3.

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References

[1] S. Chowla: The Riemann zeta and allied functions, Bull. Amer. Math. Soc., **58** (1952).

[2] S. Szidon: Theorie der trigonometrischen und orthogonalreihen, Acta Szeged, **10** (1941).