21. Some Trigonometrical Series. XIX

By Shin-ichi IZUMI
Mathematical Institute, Tokyo Metropolitan University, Tokyo
(Comm. by Z. SUETUNA, M.J.A., Feb. 13, 1956)

1. In the preceding paper [1], we have proved the following

Theorem 1.[1] *If* $p\geqq\lambda>1$, $\varepsilon>0$ *and*

$$\left(\int_0^{2\pi}|f(x+t)-f(x-t)|^p dx\right)^{1/p}=O\left(t^{1/\lambda}\Big/\left(\log\frac{1}{t}\right)^{(1+\varepsilon)/\lambda}\right),$$

then the series

$$\sum|s_n(x)-f(x)|^\lambda$$

converges almost everywhere, where $s_n(x)$ *denotes the nth partial sum of the Fourier series of* $f(x)$.

We shall here consider the case $\lambda=1$ and in fact prove the following

Theorem 2.[2] *If* $f(x)$ *is differentiable almost everywhere and*

$$(1)\qquad \left(\int_0^{2\pi}|f'(x+t)-f'(x-t)|^p dx\right)^{1/p}\leqq A\Big/\left(\log\frac{1}{t}\right)^\beta$$

where $p>1$ *and* $\beta>1$, *then the series*

$$(2)\qquad \sum|s_n(x)-f(x)|$$

converges almost everywhere.

More generally, the condition (1) may be replaced by

$$\sum_{n=1}^\infty n^{-1}\omega'_p(n^{-1})<\infty$$

where

$$\omega'_p(t)=\max_{0\leqq h\leqq t}\left(\int_0^{2\pi}|f'(x+h)-f'(x-h)|^p dx\right)^{1/p}.$$

The method of proof is similar to that of [1].

2. For the proof of Theorem 2 we need a lemma due to A. Zygmund [2]:

Lemma. *Suppose that* $p>1$ *and*

$$\left\|\sum_{\nu=m}^n \gamma_\nu e^{i\nu x}\right\|_p\leqq C$$

where $\|\ \ \|_p$ *denotes the* L^p*-norm and suppose that*

$$|\lambda_\nu|\leqq M,\quad \sum_{\nu=m}^{n-1}|\lambda_\nu-\lambda_{\nu+1}|\leqq M,$$

1) In [1], it is written that $p\geqq\lambda\geqq1$, but the case $\lambda=1$ is trivial. The assumption that "$f(t)$ is of the power series type", and its foot-note are superfluous.

2) G. Sunouchi and T. Tsuchikura remarked the author that the case $p=2$ is equivalent to a theorem of Tsuchikura [4].

then

$$\left\| \sum_{\nu=m}^{n} \gamma_\nu \lambda_\nu e^{i\nu x} \right\|_p \leqq A_p MC.$$

Let us now prove the theorem. It is sufficient to prove that the integrated series of (2)

$$(3) \qquad \sum_{n=1}^{\infty} \int_0^{2\pi} | s_n(x) - f(x) | \, dx$$

is convergent. For the sake of simplicity, let

$$(4) \qquad f(x) \sim \sum_{\nu=1}^{\infty} c_\nu e^{i\nu x},$$

then

$$f'(x) \sim \sum_{\nu=1}^{\infty} i\nu c_\nu e^{i\nu x}.$$

By the condition (1)

$$\left\| \sum_{\nu=1}^{\infty} \nu c_\nu e^{i\nu x} \sin \nu t \right\|_p \leqq A \Big/ \left(\log \frac{1}{t} \right)^\beta,$$

and by the M. Riesz theorem

$$\left\| \sum_{\nu=2^n}^{2^{n+1}-1} \nu c_\nu e^{i\nu x} \sin \nu t \right\|_p \leqq A \Big/ \left(\log \frac{1}{t} \right)^\beta.$$

If we take $t=\pi/2^{n+2}$, then we get, by the lemma,

$$\left\| \sum_{\nu=2^n}^{2^{n+1}-1} c_\nu e^{i\nu x} \right\|_p \leqq A/2^n n^\beta.$$

The estimation holds even if the lower limit of the left side summation is replaced by m such that $2^n < m < 2^{n-1}$, and its upper limit by ∞.

Thus (3) is less than

$$\sum_{n=1}^{\infty} \| s_n(x) - f(x) \|_p = \sum_{n=1}^{\infty} \sum_{\nu=2^n}^{2^{n+1}-1} \| s_\nu(x) - f(x) \|_p$$
$$\leqq A \sum_{n=1}^{\infty} 2^n/2^n n^\beta \leqq A \sum_{n=1}^{\infty} 1/n^\beta < \infty,$$

which is the required.

3. Let $f^\alpha(t)$ denote the αth derivative of $f(t)$ (cf. [3]). If $f(t)$ is given by (4) and $f^\alpha(t)$ is integrable, then

$$f^\alpha(t) = \sum_{\nu=1}^{\infty} \nu^\alpha c_\nu e^{i\nu x}.$$

Then we can prove the following

Theorem 3. *If* $0 < \alpha < 1$ *and*

$$\left(\int_0^{2\pi} | f^\alpha(x+t) - f^\alpha(x-t) |^p dt \right)^{1/p} \leqq A t^{1-\alpha} \Big/ \left(\log \frac{1}{t} \right)^\beta$$

where $p>1$ *and* $\beta>1$, *then the series* (2) *converges almost everywhere.*

References

[1] S. Izumi: Some trigonometrical series. XVI, Proc. Japan Acad., **31** (1955).
[2] A. Zygmund: Modulus of continuity of functions, Revista Math. (1952).
[3] A. Zygmund: Trigonometrical series, Warszawa (1935).
[4] T. Tsuchikura: Tôhoku Math. Journal (to appear).