20. *Some Strong Summability of Fourier Series*

By Masakiti KINUKAWA
Natural Science Division, International Christian University,
Mitaka, Tokyo

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1. The object of this paper is to find the condition of almost everywhere convergence of the series

$$(1.1)\qquad \sum_{n=1}^{\infty} |s_n(x)-f(x)|^k,$$

where $s_n(x)$ is the nth partial sum of the Fourier series of $f(x)$.

Concerning this problem, S. Izumi [2] has shown the following:

Let $p>1$, $p\geqq k>1$ and ε be any positive number. If

$$\omega_p(t)=\sup_{|u|\leqq t}\left\{\int_{-\pi}^{\pi}|f(x+u)-f(x)|^p dx\right\}^{1/p}\leqq A\, t^{1/k}\Big/\left(\log\frac{1}{t}\right)^{(1+\varepsilon)/k},$$

then the series (1.1) converges almost everywhere.

Related this theorem we shall prove some theorems.

Theorem 1. *In order that the series (1.1) converges almost everywhere, one of the following conditions is sufficient:*

$$(1.2)\qquad \sum_{\lambda=1}^{\infty}\lambda^{\gamma}[2^{\lambda/k}\omega_p(1/2^{\lambda})]^p<\infty,\ \ for\ \ 2\geqq p>k>1,\ \gamma>p/k-1,$$

$$(1.3)\qquad \sum_{\lambda=1}^{\infty}2^{\lambda}[\omega_p(1/2^{\lambda})]^p<\infty,\qquad for\ \ 2>p=k>1,$$

$$(1.4)\qquad \sum_{\lambda=1}^{\infty}\lambda\cdot 2^{\lambda}[\omega_p(1/2^{\lambda})]^p<\infty,\ \ for\ \ p=k=2.$$

2. Proof of Theorem 1.[1] We have

$$\begin{aligned} s_n(x)-f(x)&=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\sin(n+1/2)t/(2\sin t/2)\,dt\\ &=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\frac{\cos t/2}{2\sin t/2}\sin nt\,dt+\frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\varphi_x(t)\cos nt\,dt\\ &=P_n(x)+Q_n(x), \end{aligned}$$

say, where $\varphi_x(t)=\varphi(t)=f(x+t)+f(x-t)-2f(x)$, and $P_n(x)$ and $Q_n(x)$ are the nth Fourier coefficients of the functions $\varphi_x(t)\cos t/2/(2\sin t/2)$ $=\varphi_x(t)p(t)$ and $\varphi_x(t)/2$, respectively.

Let $1<p\leqq 2$ and p' be its conjugate, then by the Hausdorff-Young inequality, we get[2]

$$\left\{\sum_{n=1}^{\infty}|P_n(x)\sin nh|^{p'}\right\}^{p/p'}\leqq A\left\{\int_0^{\pi}|\varphi(t+h)p(t+h)-\varphi(t-h)p(t-h)|^p dt\right\}$$

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1) Cf. N. Matsuyama [3].

2) We denote by A an absolute constant, which is not necessarily the same in different occurrences.

$$\leqq A\left\{\int_0^\pi |\varphi(t+h)-\varphi(t-h)|^p |p(t+h)|^p\,dt + \int_0^\pi |p(t+h)-p(t-h)|^p|\varphi(t-h)|^p dt\right\}$$

$$= A\left\{I_1(x)+I_2(x)\right\}.$$

Then we have

$$\int_{-\pi}^\pi I_1(x)\,dx = \int_{-\pi}^\pi dx\int_0^\pi |\varphi_x(t+h)-\varphi_x(t-h)|^p|p(t+h)|^p\,dt$$

$$\leqq A\int_0^\pi \omega_p^p(h)|p(t+h)|^p\,dt \leqq A\omega_p^p(h)\int_0^\pi (t+h)^{-p}\,dt$$

$$\leqq A\omega_p^p(h)h^{1-p},$$

and

$$\int_{-\pi}^\pi I_2(x)\,dx \leqq A\int_{-\pi}^\pi dx\left\{\int_0^h + \int_h^\pi\right\}dt = A(B_1+B_2),$$

where

$$B_1 = \int_{-\pi}^\pi dx\int_0^h |p(t+2h)-p(t)|^p|\varphi_x(t)|^p\,dt \leqq A\int_{-\pi}^\pi dx\int_0^h |\varphi_x(t)|^p/t^p\,dt$$

$$\leqq A\int_0^h \omega_p^p(t)\,t^{-p}\,dt$$

and

$$B_2 = \int_{-\pi}^\pi dx\int_0^{\pi-h} |p(t+2h)-p(t)|^p|\varphi_x(t)|^p\,dt$$

$$\leqq A\left\{\int_0^h \omega_p^p(t)\,t^{-p}\,dt + \int_{-\pi}^\pi dx\int_h^\pi |p(t+2h)-p(t)|^p|\varphi_x(t)|^p\,dt\right\}$$

$$\leqq A\left\{\int_0^h \omega_p^p(t)\,t^{-p}\,dt + Ah^p\int_h^\pi \omega_p^p(t)\,t^{-2p}\,dt\right\}.$$

Collecting above estimations, we get

$$\int_{-\pi}^\pi \left\{\sum_{n=1}^\infty P_n(x)\sin nh|^{p'}\right\}^{p/p'}dx \leqq \int_{-\pi}^\pi I_1(x)\,dx + \int_{-\pi}^\pi I_2(x)\,dx$$

$$\leqq A\left\{\omega_p^p(h)\,h^{1-p} + \int_0^h \omega_p^p(t)\,t^{-p}\,dt + h^p\int_h^\pi \omega_p^p(t)\,t^{-2p}\,dt\right\}.$$

Let $h=\pi/2^{(\lambda+1)}$, then

(2.1)
$$\int_{-\pi}^\pi \left\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda} |P_n(x)|^{p'}\right\}^{p/p'}dx$$

$$\leqq A\left\{\omega_p^p(h)h^{1-p} + \int_0^h \omega_p^p(t)t^{-p}dt + h^p\int_h^\pi \omega_p^p(t)t^{-2p}\,dt\right\}.$$

For the proof of Theorem 1, it is sufficient to show that the series

$$\sum_{n=1}^\infty \int_{-\pi}^\pi |P_n(x)|^k\,dx$$

is convergent, since the corresponding series containing $Q_n(x)$ is estimated similarly. If we suppose $2\geqq p>k>0$, then $p'/k>1$, where $1/p+1/p'=1$.

Hence, by the Hölder inequality and (2.1), we have

$$\sum_{n=2}^{\infty}\int_{-\pi}^{\pi}|P_n(x)|^k\,dx=\sum_{\lambda=1}^{\infty}\sum_{n=2^{\lambda-1}+1}^{2^\lambda}\int_{-\pi}^{\pi}|P_n(x)|^k\,dx$$
$$\leqq\sum_{\lambda=1}^{\infty}\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}}^{2^\lambda}|P_n(x)|^{p'}\Big\}^{k/p'}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}1\Big\}^{1-k/p'}dx$$
$$\leqq A\sum_{\lambda=1}^{\infty}2^{\lambda(1-k/p')}\Big[\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}|P_n(x)|^{p'}\Big\}^{p/p'}dx\Big]^{k/p}$$
$$\leqq A\sum_{\lambda=1}^{\infty}2^{\lambda(1-k/p')}\Big\{\omega_p^k(h)\,h^{(1-p)k/p}+\Big(\int_0^h\omega_p^p(t)\,t^{-p}\,dt\Big)^{k/p}+h^k\Big(\int_h^{\pi}\omega_p^p(t)\,t^{-2p}\,dt\Big)^{k/p}\Big\}$$
$$=A(S_1^{(k)}+S_2^{(k)}+S_3^{(k)}),$$

where $S_1^{(k)}$ is finite by the assumption (1.2). By the Hölder inequality,

$$S_2^{(k)}\leqq A\Big\{\sum_{\lambda=1}^{\infty}(\lambda+1)^{-\gamma k/(p-k)}\Big\}^{1-k/p}\Big\{\sum_{\lambda=1}^{\infty}2^{\lambda(1-k/p')p/k}(\lambda+1)^{\gamma}\int_0^h\omega_p^p(t)\,t^{-p}\,dt\Big\}^{k/p}$$
$$\leqq A\Big\{\sum_{\nu=1}^{\infty}\int_{\pi/2^{\nu+1}}^{\pi/2^\nu}\omega_p^p(t)\,t^{-p}\,dt\sum_{\lambda=1}^{\nu-1}(\lambda+1)^{\gamma}2^{\lambda(1-k/p')p/k}\Big\}^{k/p}$$
$$\leqq A\Big\{\sum_{\nu=1}^{\infty}\nu^{\gamma}[2^{\nu/k}\omega_p(1/2^\nu)]^p\Big\}^{k/p}<\infty,$$

since $\gamma k/(p-k)>1$. We have also

$$S_3^{(k)}\leqq A\Big\{\sum_{\lambda=1}^{\infty}2^{\lambda(1-k/p')p/k}2^{-\lambda p}(\lambda+1)^{\gamma}\sum_{\nu=0}^{\lambda}\int_{\pi/2^{\nu+1}}^{\pi/2^\nu}\omega_p^p(t)\,t^{-2p}\,dt\Big\}^{k/p}$$
$$\leqq A\Big\{\sum_{\nu=0}^{\infty}\int_{\pi/2^{\nu+1}}^{\pi/2^\nu}\omega_p^p(t)\,t^{-2p}\,dt\sum_{\lambda=\nu}^{\infty}2^{\lambda(p/k+1-2p)}(\lambda+1)^{\gamma}\Big\}^{k/p}$$
$$\leqq A\Big\{\sum_{\nu=0}^{\infty}2^{\nu(2p-1)}\omega_p^p(1/2^\nu)\,2^{\nu(p/k+1-2p)}(\nu+1)^{\gamma}\Big\}^{k/p}$$
$$\leqq A\Big\{\sum_{\nu=0}^{\infty}(\nu+1)^{\gamma}[2^{\nu/k}\omega_p(1/2^\nu)]^p\Big\}^{k/p}<\infty,$$

since $p/k+1-2p<0$. Thus we get the theorem for the case $2\geqq p>k>1$.

For the case $2\geqq p=k>1$, we have, since $2\geqq p>1$ and so $p'/p\geqq 1$,

$$\sum_{n=2}^{\infty}\int_{-\pi}^{\pi}|P_n(x)|^k\,dx=\sum_{n=2}^{\infty}\int_{-\pi}^{\pi}|P_n(x)|^p\,dx=\sum_{\lambda=1}^{\infty}\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}|P_n(x)|^p\Big\}dx$$
$$\leqq\sum_{\lambda=1}^{\infty}\int_{-\pi}^{\pi}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}|P_n(x)|^{p'}\Big\}^{p/p'}\Big\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda}1\Big\}^{1-p/p'}dx$$
$$\leqq A\sum_{\lambda=1}^{\infty}2^{\lambda(1-p/p')}\Big\{\omega_p^p(h)\,h^{1-p}+\int_0^h\omega_p^p(t)\,t^{-p}\,dt+h^p\int_h^{\pi}\omega_p^p(t)\,t^{-2p}\,dt\Big\}$$

$$=A\left\{S_1^{(p)}+S_2^{(p)}+S_3^{(p)}\right\},$$

where

$$S_1^{(p)}\leqq A\sum_{\lambda=1}^{\infty}2^{\lambda}\omega_p(1/2^{\lambda})<\infty,$$

$$S_2^{(p)}\leqq A\sum_{\nu=1}^{\infty}\int_{\pi/2^{\nu+1}}^{\pi/2^{\nu}}\omega_p^p(t)\,t^{-p}\,dt\sum_{\lambda=1}^{\nu}2^{\lambda(2-p)}$$

$$\leqq A\begin{cases}\sum_{\nu=1}^{\infty}2^{\nu(2-p)}2^{\nu(p-1)}\omega_p^p(1/2^{\nu}) & (p\neq 2)\\ \sum_{\nu=1}^{\infty}\nu 2^{\nu}\,\omega_p^p(1/2^{\nu}) & (p=2)\end{cases}$$

$$\leqq A\begin{cases}\sum_{\nu=1}^{\infty}2^{\nu}\,\omega_p^p(1/2^{\nu}) & (p\neq 2)\\ \sum_{\nu=1}^{\infty}\nu 2^{\nu}\,\omega_p^p(1/2^{\nu}) & (p=2)\end{cases}$$

and

$$S_3^{(p)}\leqq A\sum_{\lambda=1}^{\infty}2^{\lambda(2-p-p)}\sum_{\nu=0}^{\lambda}\int_{\pi/2^{\nu+1}}^{\pi/2^{\nu}}\omega_p^p(t)\,t^{-2p}\,dt\leqq A\sum_{\lambda=1}^{\infty}2^{\nu}\omega_p^p(1/2^{\nu})<\infty.$$

Hence we get Theorem 1.

3. Using the above argument, we can easily get the following[3)]

Theorem 2. *In order that the series*

$$\sum_{n=1}^{\infty}n^{\beta}\,|s_n(x)-f(x)|^k$$

converges almost everywhere, one of the following conditions is sufficient:

(3.1) $\sum_{\lambda=1}^{\infty}\lambda^{\gamma}[2^{\lambda(1+\beta)/k}\omega_p(1/2^{\lambda})]^p<\infty$, *for* $2\geqq p>k>1+\beta$, $\gamma>p/k-1,\ p>1,\ k>0$

(3.2) $\sum_{\lambda=1}^{\infty}2^{\lambda(1+\beta)}[\omega_p(1/2^{\lambda})]^p<\infty$, *for* $2>p=k>1+\beta$,

(3.3) $\sum_{\lambda=1}^{\infty}\lambda\cdot 2^{\lambda(1+\beta)}[\omega_p(1/2^{\lambda})]^p<\infty$, *for* $2=p=k,\ 1>\beta$.

Especially, if we consider the case $\beta=-1$ in Theorem 2, then we find the condition for the almost everywhere convergence of the series

$$\sum_{n=1}^{\infty}|s_n(x)-f(x)|^k/n,$$

which relates a theorem due to T. Tsuchikura [4].

References

[1] S. Izumi: Some trigonometrical series, VI, Tôhoku Math. Jour., **5**, 290–295 (1954).
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[3] N. Matsuyama: On the $|C|$-summability of the Fourier series, Tôhoku Math. Jour., **2**, 51–56 (1950).
[4] T. Tsuchikura: Convergence character of Fourier series at a point, Mathematica Japonicae, **1** (1949).

3) Cf. S. Izumi [1].