53. *Evans-Selberg's Theorem on Abstract Riemann Surfaces with Positive Boundaries. I*

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Let R^* be a Riemann surface with a positive boundary and let $\{R_n\}$ $(n=0,1,2,\cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. Let $N(z,p)$ be a positive function in R harmonic in R except one point $p\in R$ such that $N(z,p)=0$ on ∂R_0, $N(z,p)+\log|z-p|$ is harmonic in a neighbourhood of p and the $*$-Dirichlet integral taken over R is minimal, where the $*$-Dirichlet integral is taken with respect to $N(z,p)+\log|z-p|$ in a neighbourhood of p. It is easily seen that such $N(z,p)$ is uniquely determined and $\int_{\partial R_0}\frac{\partial N(z,p)}{\partial n}ds=2\pi$. As in the case when R^* is a Riemann surface with a null-boundary, we define the ideal boundary point, by making use of $N(z,p)$, that is, if $\{p_i\}$ is a sequence of points in R having no accumulation point in $R+\partial R_0$, for which the corresponding functions $N(z,p_i)$ $(i=1,2,\cdots)$ converge uniformly in every compact set of R, we say that $\{p_i\}$ is a fundamental sequence. Two fundamental sequences are equivalent, if and only if, their corresponding sequences of functions have the same limit function. The equivalent sequences are made to correspond to an ideal boundary point. The set of all the ideal boundary points will be denoted by B and the set $R+B$, by $\overline{R}$. The domain of definition of $N(z,p)$ may now be extended by writing $N(z,p)=\lim_{i=\infty}N(z,p_i)$ $(z\in R,\ p\in B)$, where $\{p_i\}$ is any fundamental sequence. For p in B, the flux of $N(z,p)$ along ∂R_0 is also 2π. The distance between two points p_1 and p_2 of $\overline{R}$ is defined by

$$\delta(p_1,\,p_2)=\sup_{z\in R_1-R_0}\left|\frac{N(z,\,p_1)}{1+N(z,\,p_1)}-\frac{N(z,\,p_2)}{1+N(z,\,p_2)}\right|.$$

The topology induced by this metric is homeomorphic to the original topology in R and we see easily that $R-R_1+\partial R_1+B$ and B are closed and compact.

At first, we have the following

Lemma 1. Put $N^M(z,p)=\min[M,N(z,p)]$. *Then the Dirichlet integral of* $N^M(z,p)$ *over* R *satisfies*

$$D(N^M(z,p))\leqq 2\pi M,\qquad M\geqq 0,$$

for every point of $\overline{R}$.

In what follows, in order to introduce the harmonicity or superharmonicity in $\overline{R}$ (not only in R), we make some preparations as follows.

1. *Capacity and the Equilibrium Potential of Relatively Closed Sets in R.*

Let F be a compact or non compact relatively closed set in R having no common point with R_1. Denote by $\omega_n(z)$ a harmonic function in R_n-R_0-F such that $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=1$ on F except possibly a subset of F of capacity zero and $\frac{\partial \omega_n(z)}{\partial n}=0$ on ∂R_n-F. Then it is proved that $\omega_n(z)$ converges to $\omega_F(z)$ in mean. $\omega_F(z)$ and the Dirichlet integral $D(\omega_F(z))=\int_{\partial R_0}\frac{\partial \omega_F(z)}{\partial n}ds$ are called the equilibrium potential and the capacity of F respectively. We have the following

Theorem 1. 1) *If $F_n\uparrow F$, then $\omega_{F_n}(z)\uparrow\omega_F(z)$ and* $\mathrm{Cap}(F_n)\uparrow\mathrm{Cap}(F)$.

2) *Let G_ε be the domain $G_\varepsilon=E[z\in R\colon \omega_F(z)>1-\varepsilon]$ and let $\omega_{G_\varepsilon}(z)$ be the equilibrium potential of G_ε. Then*

$$\omega_F(z)=(1-\varepsilon)\omega_{G_\varepsilon}(z),$$

where ε is a positive number such that $0\leqq\varepsilon\leqq 1$.

3) *Let ∂G_ε be the niveau curve of $\omega_F(z)$ with height $1-\varepsilon$. Then there exists a set H in the interval $[0,1]$ such that* $\mathrm{mes}\,H=0$ *and*

$$\mathrm{Cap}(F)=\int_{\partial R_0}\frac{\partial \omega_F(z)}{\partial n}ds=\int_{\partial G_\varepsilon}\frac{\partial \omega_F(z)}{\partial n}ds$$

for $1-\varepsilon\notin H$.

In the present paper, we consider only positive continuous function $U(z)$ such that $U(z)=0$ on ∂R_0 and $D(U^M(z))<\infty$ for every M, where $U^M(z)=\min[M, U(z)]$.

2. *Regular Domains.* Let F be a compact or non compact domain in R and let $\omega_F(z)$ be its equilibrium potential. If $\int_{\partial F}\frac{\partial \omega_F(z)}{\partial n}ds=\int_{\partial R_0}\frac{\partial \omega_F(z)}{\partial n}ds$, F is called a regular domain. We see at once by 3) of Theorem 1 that there exists a sequence of regular domains $G_\varepsilon=E[z\in R\colon \omega_F(z)\geqq 1-\varepsilon]$ which we call the *regular domains generated by the equilibrium potential*, containing F of capacity positive and that any compact closed domain with analytic relative boundaries is always regular.

Suppose a continuous function $U(z)$ in R such that $U(z)=0$ on ∂R_0, $D(U^M(z))<\infty$ and a regular domain D. Let $U_D^M(z)$ be a harmonic function in $R-D$ such that $U_D^M(z)=U^M(z)$ on $\partial R_0+\partial D$ and $U_D^M(z)$ has the minimal Dirichlet integral over $R-D$. Then evidently, $U_D^M(z)$ is determined uniquely. Put $U_D(z)=\lim_{M=\infty}U_D^M(z)$. On the other hand, let $N^D(z,p)$ be a function in $R-D$ such that $N^D(z,p)$ is harmonic in $R-D$ except p where $N^D(z,p)+\log|z-p|$ is harmonic, $N^D(z,p)=0$ on $\partial R_0+\partial D$ and $N^D(z,p)$ has the minimal $*$-Dirichlet integral, where it is taken with respect to $N^D(z,p)+\log|z-p|$ in a neighbourhood of p. Then we have the following

Theorem 2. $$U_D(p)=\frac{1}{2\pi}\int_{\partial D}U(z)\frac{\partial N^D(z,p)}{\partial n}ds.$$

3. *Harmonicity and Superharmonicity in* $\overline{R}$. For any compact or non compact regular domain D, if $U(z)=U_D(z)$ or $\geqq U_D(z)$, we say that $U(z)$ is *harmonic* or *superharmonic* in $\overline{R}$ respectively. Then we have the following

Theorem 3. $N(z,p)$ *is superharmonic in* $\overline{R}$, *more generally* $\int N(z,p_a)d\mu(p_a)$ *is superharmonic in* $\overline{R}$, *where* $\mu\geqq 0$.

Let $U(z)$ be a positive harmonic function in R and superharmonic in $\overline{R}$ vanishing on ∂R_0 and let D be a relatively closed set in R of capacity positive. If D is regular, we define $U_D(z)$ as in Theorem 2 and if D is not regular, we define $U_D(z)$ as follows: suppose that $\{D_n\}$ is a sequence of decreasing regular domains generated by the equilibrium potential $\omega_D(z)$ of D. Let $U^M_{D_n}(z)$ be a harmonic function in $R-D_n$ such that $U^M_{D_n}(z)=U^M(z)$ on $\partial D_n+\partial R_0$ and $U^M_{D_n}(z)$ has the minimal Dirichlet integral over $R-D_n$. Then by the superharmonicity of $U^M(z)$ (which is easily verified as in space by the superharmonic of $U(z)$), we have $U^M_{D_n}(z)\leqq U^M(z)$ and $U_{D_n}(U^M_{D_{n+i}}(z))=U^M_{D_{n+i}}(z)$ for $D_n\supset D_{n+i}$. Let M tend to ∞. Then we have at once $U_{D_n}(U_{D_{n+i}}(z))=U_{D_{n+i}}(z)$. Hence $U_{D_n}(z)$ is decreasing as D_{D_n} decreases. We define $U_D(z)$ by $\lim\limits_{n=\infty} U_{D_n}(z)$. Then we have the following

Theorem 4. If $U(z)$ *and* $V(z)$ *are positive,* $U(z)=V(z)=0$ *on* ∂R_0 *and superharmonic in* $\overline{R}$, *then*

1) $U_D(z)\leqq U(z)$.

2) If $U(z)\geqq V(z)$, $U_D(z)\geqq V_D(z)$.

3) $U_D(z)+V_D(z)={}_D(U(z)+V(z))$.

4) If $C\geqq 0$, $(CU_D(z))={}_D(CU(z))$.

5) For D_1 *and* D_2, $U_{D_1+D_2}(z)\leqq U_{D_1}(z)+U_{D_2}(z)$.

6) If $D_1\supset D_2$, *then* ${}_{D_1}(U_{D_2}(z))=U_{D_2}(z)$ *and* $U_{D_1}(z)\geqq U_{D_2}(z)$.

7) Let $\{D_n\}$ *be an increasing sequence of regular domains such that* $D_n=E[z\in R\colon\ \omega_D(z)\geqq 1-\varepsilon_n]$ *and* $D_n\uparrow D_0$, *where* $D_0=E[z\in R\colon \omega_D(z)\geqq 1-\varepsilon_0]$ *is also regular domain. Then* $U_{D_n}(z)\uparrow U_{D_0}(z)$.

4. *Integral Representation of Superharmonic Functions in* $\overline{R}$.

Let A be a δ-closed subset of B (closed with respect to δ-metric). Put $A_n=E\left[z\in\overline{R}\colon\ \delta(z,A)\leqq\frac{1}{n}\right]$. Then A_n is a relatively closed set and $\bigcap\limits_n A_n=A$. Let $\omega_{A_n}(z)$ be the equilibrium potential of A_n. Then we see that $\omega_{A_n}(z)$ converges to $\omega_A(z)$ in mean. $\omega_A(z)$ is called the equilibrium potential of A and $D(\omega_A(z))=\int\limits_{\partial R_0}\frac{\partial\omega_A(z)}{\partial n}ds$ is called capacity. For δ-closed subset A of B, we define $U_A(z)$ by $\lim\limits_{n=\infty}U_{G_n}(z)$, where $G_n=E[z\in R\colon\ \omega_{A_n}(z)\geqq 1-\varepsilon_n]$ and $\lim\limits_{n=\infty}\varepsilon_n=0$. By definition $G_n\supset A$. Put $\bigcap\limits_{n=1}^{\infty}G_n=A^*$ and call A^* the *capacity closure of A*. Then we have the following

Theorem 5. 1) Assertions of Theorem 4 hold for $U_A(z)$.

2) $$U_A(z)=\int_{A^*} N(z,p)d\mu(p)$$

for all points z *in* R. *The total mass* $\mu(A^*)$ *is equal to* $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U_A(z)}{\partial n}ds$.

2') $$\omega_A(z)=\int_A N(z,p)d\mu(p).$$

3) $$U(z)=\int_B N(z,p)d\mu(p).$$

5. *Minimal Functions. Let* $U(z)$ *be a function which is harmonic in* R *and superharmonic in* $\overline{R}$. *If* $U(z)\geqq V(z)$ *implies* $V(z)=kU(z)$ $(k\leqq 1)$ *for every function* $V(z)$ *such that both* $V(z)$ *and* $U(z)-V(z)$ *are harmonic and superharmonic in* $\overline{R}$, $U(z)$ *is called a minimal function.* We shall obtain characteristics of minimal functions.

Theorem 6. Suppose that $U(z)$ *is positive and minimal. Let* A *be a* δ*-closed set of* B. *If now the following relation of the form holds*

$$U(z)\geqq U_A(z)=\int_{A^*} N(z,p)\,d\mu(p)>0, \qquad z\in R,$$

then $U(z)=\left(\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds\right)N(z,q)$, *where* q *is a point of* A^*.

Corollary. Every minimal function in $\overline{R}$ *is a positive multiple of some* $N(z,q)$ $(q\in B)$.

Put $A=q$ and define the function $\psi(q)$ for q in B as $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial N_q(z,q)}{\partial n}ds$.

Then we have

Theorem 7. 1) $\psi(q)$ *has only two possible values* 1 *and* 0.

2) *Denoting by* B_0 *the set of points of* B *for which* $\psi(q)=0$, B_0 *is void or an* F_σ.

We consider B_1 where B_1 is the set of points of which $\psi(q)=1$. Then

Theorem 8. 1) *If* $U(z)$ *is given by* $\frac{1}{2\pi}\int_{B_0}N(z,p)d\mu(p)$, *then* $U_{B_0}(z)=0$ *and* $U(z)=\int_{B_1}N(z,p)d\mu(p)$ *for every harmonic in* R *and superharmonic function* $U(z)$ *in* $\overline{R}$.

2) $$\mathrm{Cap}(B_0)=0.$$

Hence every positive mass distribution on B_0 *can be replaced by that on* B_1. But the present author can not prove the uniqueness of mass distribution. In what follows, we shall prove usefull properties of points in B_1.

Theorem 9. 1) $N(z,p)$ *is minimal or not according to* $p\in B_1$ *or not.*

2) Let $V_m(p)=E[z\in R: N(z,p)\geqq m]$ *and* $v_n(p)=E\Big[z\in \overline{R}: \delta(z,p)\leqq\frac{1}{n}\Big]$. *Then if* p *is a minimal point,*

$$N_{V_m(p)}(z,p)=N(z,p)$$

for every m *less than* $\sup_{z\in R} N(z,p)=M'$.

2′) There exists a set H *in* $[0, M']$ *such that* $\operatorname{mes} E=0$ *and that if* $m\notin E$, *then* $\int\limits_{\partial V_m(p)}\frac{\partial N(z,p)}{\partial n}=2\pi$, *for minimal* $N(z,p)$ *or* $N(z,p)$ *with* $p\in R$.

2″) For every $V_m(p)$, *there exists a number* n *such that* $V_m(p)\supset(v_n(p)\cap R)$, *for minimal* $N(z,p)$ *or* $N(z,p)$ *with* $p\in R$.

6. *The Function* $N(z,p)$. Assume that p and q are contained in R. Let $N_n(z,p)$ and $N_n(z,q)$ be functions in R_n-R_0 such that $N_n(z,p)$ and $N_n(z,q)$ are harmonic in R_n-R_0 except p and q respectively where $N_n(z,p)$ and $N_n(z,q)$ have logarithmic singularities and $\frac{\partial N_n(z,p)}{\partial n}=\frac{\partial N_n(z,q)}{\partial n}=0$ on ∂R_n. Then we have by Green's formula $N_n(q,p)=N_n(p,q)$. Since $N_n(z,p)\to N(z,p)$ as $n\to\infty$, we have $N(q,p)=N(p,q)$ by letting $n\to\infty$. Let $\{q_i\}$ be a fundamental sequence determining a point $q\in B$. Then, since $N(z,q_i)$ tends to $N(z,q)$ at every point z of R, $N(p,q_i)=N(q_i,p)$ implies that $N(z,p)$ has limit as z tends to q. This limit is denoted by $N(q,p)$. Hence if $p\in R$, $N(z,p)$ has limit as $z(\in\overline{R})$ tends to q with respect to δ-metric. We define the value $N(z,p)$ at q by this limit. Therefore, if $p\in R$, then $N(z,p)$ is defined at every point z of $\overline{R}$ and $N(z,p)$ is δ-continuous, except $z=p$. In what follows, we shall study the case when $p\in B$.

Suppose that p is minimal. Then by 2) of Theorem 9 $\frac{N(z,p)}{m}$ can be considered as the equilibrium potential of $V_m(p)$ for every m less than $\sup_{z\in R} N(z,p)$. Let $V_m(p)$ be regular. $V_m(p)$ may consist of at most enumerably infinite number of domains D_k $(k=1,2,\cdots)$. $N(z,p)$ can not be a constant in every D_k, hence there exists a constant m_k depending on D_k such that D_k contains some components D' of $V_{m'}(p)$. By 2) of Theorem 9 $N(z,p)$ can be considered as the equilibrium potential of D' with respect to D_k, that is, $N(z,p)-m=0$ on ∂D_k, $N(z,p)=m'-m$ on $\partial D'$ and $N(z,p)$ has the minimal Dirichlet integral taken over D_k-D'. By the regularity of $V_m(p)$

$$\lim_{n=\infty}\int\limits_{\partial V_m(p)\cap(R_n-R_0)}\frac{\partial N_n(z,p)}{\partial n}ds=\int\limits_{\partial V_m(p)}\frac{\partial N(z,p)}{\partial n}ds, \tag{1}$$

where $N_n(z,p)$ is harmonic in $R_n-R_0-V_{m'}(p)$ $(m'>m)$ $N_n(z,p)=0$ on

∂R_0 and $N_n(z, p)=m'$ on $\partial V_{m'}(p)$ and $\dfrac{\partial N_n(z, p)}{\partial n}=0$ on $\partial R_n - V_m(p)$. On the other hand, by Fatou's lemma

$$\lim_{n=\infty}\int_{\partial D_k\cap(R_n-R_0)}\frac{\partial N_n(z, p)}{\partial n}ds \geqq \int_{\partial D_k}\frac{\partial N(z, p)}{\partial n}ds.$$

Hence by (1), for every domain D_k,

$$\lim_{n=\infty}\int_{\partial D_k\cap(R_n-R_0)}\frac{\partial N_n(z, p)}{\partial n}ds = \int_{\partial D_k}\frac{\partial N(z, p)}{\partial n}ds.$$

Let $N_{D,n}(\zeta, z)$ be a function in $D_k\cap(R_n-R_0)$ such that $N_{D,n}(\zeta, z)=0$ on $\partial D_k\cap(R_n-R_0)+\partial R_0$, $\dfrac{\partial N_{D,n}(\zeta, z)}{\partial n}=0$ on $\partial R_n\cap D_k$ and $N_{D,n}(\zeta, z)$ is harmonic in $D_k\cap(R_n-R_0)$ except p where $N_{D,n}(\zeta, z)$ has a logarithmic singularity. Then there exists a constant L such that

$$L(N_n(\zeta, z)-m)\geqq N_{D,n}(\zeta, z) \text{ in } (D_k\cap(R_n-R_0))-V(z),$$

where $V(z)$ is a suitable neighbourhood of z. Hence

$$\lim_{n=\infty}\int_{\partial D_k\cap(R_n-R_0)}\frac{\partial N_{D,n}(\zeta, z)}{\partial n}ds = \int_{\partial D_k}\lim_{n=\infty}\frac{N_{D,n}(\zeta, z)}{\partial n}ds. \tag{2}$$

We call $N_D(\zeta, z)=\lim\limits_{n=\infty} N_{D,n}(\zeta, z)$ the Green's function of D_k with pole at z. Apply the Green's formula to $N(\zeta, q_i)$ and $N_D(\zeta, z)$. Then by (2) we have

$$\frac{1}{2\pi}\int_{\partial D_k}N(\zeta, q_i)\frac{\partial N_D(\zeta, z)}{\partial n}ds < \text{ or } = N(z, q_i)$$

according to $q_i \in D_k$ or not. Let $i\to\infty$. Then by Fatou's lemma

$$\frac{1}{2\pi}\int_{\partial D_k}N(\zeta, q)\frac{\partial N_D(\zeta, z)}{\partial n}ds \leqq N(z, q). \tag{3}$$

Let $N^M_{D,n}(z, q)$ be a harmonic function in $D_k\cap(R_n-R_0)$ such that $N^M_{D,n}(z, q)=N^M(z, q)$ on $\partial D_k\cap(R_n-R_0)$ and $\dfrac{\partial N^M_{D,n}(z, q)}{\partial n}=0$ on $\partial R_n\cap D_k$. Then $\sum\limits_k D(N^M_{D,k,n}(z, q))\leqq 2\pi M$ by Dirichlet principle. Let $n\to\infty$. Then $N^M_{D,k,n}(z, q)$ tends to $N^M_{D,k}(z, q)$ in every domain D_k and the sum of Dirichlet integrals of $N^M_{D,k}(z, q)$ over D_k is less than $2\pi M$. For simplicity, we denote by $N^M_{V_m(p)}(z, q)$ the function being equal to $N^M_{D,k}(z, q)$ in every domain D_k. Let $V_{m'}(p)$ be a regular domain such that $m'>m$. Then we have by Green's formula

$$\int_{\partial V_m(p)}N^M_{V_m(p)}(z, q)\frac{\partial N(z, p)}{\partial n}ds = \int_{\partial V_{m'}(p)}N^M_{V_m(p)}(z, q)\frac{\partial N(z, p)}{\partial n}ds.$$

By letting $M\to\infty$ and by (3)

$$N_{V_m(p)}(p, q)=\frac{1}{2\pi}\int_{\partial V_m(p)}N(z, q)\frac{\partial N(z, p)}{\partial n}ds \leqq \frac{1}{2\pi}\int_{\partial V_{m'}(p)}N(z, q)\frac{\partial N(z, p)}{\partial n}ds$$
$$= N_{V_{m'}(p)}(p, q).$$