74. Contributions to the Theory of Semi-groups. III

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Recently, the present author [2, 3] and Miss Y. Miyanaga [4] showed that some well-known theorems on compact abelian semi-groups were generalized to homogroups.

Our purpose is to generalize a theorem on compact abelian semi-groups to arbitrary homogroups.

A semi-group S is called a *homogroup*, if

1) S contains an idempotent e,

2) for every element x of S, there are two elements x' and x'' such that $xx' = e = x''x$,

3) for every element x of S, $xe = ex$.

Some writers have proved that any simple compact abelian semi-group is a compact group. We shall prove the following

Theorem 1. Any simple homogroup is a group.

For a semi-group with topology, we have

Theorem 2. Any simple compact homogroup is a compact group.

As any compact abelian semi-group is homogroup (see K. Iséki [1]), by Theorem 1, we have the following

Theorem 3. Any simple compact abelian semi-group is compact group.

The proof of Theorem 1. To prove it, we shall use the following lemma on simple semi-groups by D. Rees [5].

Lemma. A necessary and sufficient condition for semi-group S to be simple is that $SxS = S$ for every element x of S.

Let S be a simple homogroup, and e the idempotent stated in the condition 1), then, by the simplicity of S and Lemma, we have

$$SeS = S.$$

By the condition 3), we have

$$(*) \qquad\qquad S \cdot Se = S.$$

$S^2 (= S \cdot S)$ is a two-sided ideal of S and contains the idempotent e. If $S^2 = e$, then, by $(*)$, we have

$$e^2 = e = S.$$

Hence S is a group. On the other hand, if $S^2 = S$, $(*)$ implies

$$Se = S.$$

By a well-known theorem (see G. Thierrin [6]), the set Se is a group ideal of S. Therefore S is a group. The proof is complete.

Therefore, we have the following

Theorem 4. The closed minimal ideal of a compact abelian group is a compact group.

References

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