95. A Remark on the Ideal Boundary of a Riemann Surface

By Shin'ichi MORI and Minoru OTA
Ritsumeikan University and Aichi University
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Let W be an open Riemann surface, and BD the class of piecewise smooth functions f defined on W and bounded on it having a finite Dirichlet integral $D[f]$. We say that a sequence $\{f_n\}$, $f_n \in BD$, converges to f in BD if $\{f_n\}$ is uniformly bounded and $f_n \to f$ uniformly on every compact subset of W while $D[f_n - f] \to 0$.

The class of BD functions with compact carriers forms an ideal in the ring BD. If we denote by $\overline{K}$ these functions which are limits in BD of sequences from K, we have the decomposition of $f \in BD$ as follows:

$$f = \varphi + u, \qquad \varphi \in \overline{K}, \quad u \in HBD,$$

where HBD is the class of harmonic functions in BD.

To make use of the theory of normed ring, we introduce in BD a new norm given by

$$\|f\| = \sup_W |f| + \sqrt{D[f]}.$$

Since HBD is complete in this norm, the completion of $\overline{K}$ completes BD. The notation $\overline{K}$ is again used for the completed $\overline{K}$. This completed BD is a normed ring A and the set of the maximal ideals constructs a compact Hausdorff space W^* containing W as a dense subset. $f \in A$ can be represented as a continuous function on W^* and it is equivalent to $f \in M$ that f equals to zero on a point M of W^*.

The maximal ideals, which contain K, form a closed non-dense subset Γ of W that does not correspond to the inner points of W. And we regard it as the ideal boundary of W (Royden [4]). Following the statement of Royden [4] the set Δ of the maximal ideals containing $\overline{K}$ is named the harmonic boundary of W. Δ is a closed subset of Γ, which disappears in the parabolic case.

The existence of $\Gamma - \Delta$ for the hyperbolic case is known in a special case. To see this, we first observe the behaviour of Green's function $g(p, q)$ of W on Δ. By Royden [3]

$$\overline{g}(p, q) = \min[l, g(p, q)] \in \overline{K}.$$

This shows that $\overline{g}(p, q)$ represented as a function of W^* is $\overline{g}(M, q) = 0$ for $M \in \Delta$.

Now we note that a separation of W into disjoint parts by a finite number of compact curves also separates Γ. We assume that W has an end W' bounded by a finite number of compact curves and the ideal boundary of W' is harmonic measure zero with respect to W', then by Myrberg [2] we have

$$\inf_{p\in W'} \bar{g}(p, q)>0.$$

From a lemma concerning the normed ring (Gelfand [1]) we can conclude that $\bar{g}(M, q)$ is positive for the maximal ideals $M \in \Gamma$ which are associated to W'.

This means that $M \bar{\in} \Delta$ and so $M \in \Gamma-\Delta$. Therefore we can see that in this case $\Gamma-\Delta$ is not null.

We prove the following

Theorem. The harmonic function $u \in HBD$ attains its maximum and minimum on Δ.

Proof. It is evident that u attains its maximum and minimum on Γ and we want to prove it for the subset Δ. Put

$$\inf_{W} u=\lambda,$$

then $u-\lambda>0$ on W and there is a sequence of points $\{p_n\}$ $(n=1, 2,\cdots)$ such that $u(p_n)-\lambda \to 0$.

Let $h \in HBD$ be arbitrary and Ω_n $(n=1,2,\cdots)$ an exhaustion of W, then we have the decomposition

$$(u-\lambda)h=\varphi_n+v_n,$$

where $\varphi_n \in K$ and v_n is harmonic in Ω_n and equals to $(u-\lambda)h$ on $W-\Omega_n$. Since $|h| \leq C$ for a certain C, we have the following inequality,

$$-C(u-\lambda) \leq (u-\lambda)h \leq C(u-\lambda).$$

This holds particularly on the boundary of Ω_n and then

$$-C(u-\lambda) \leq v_n \leq C(u-\lambda).$$

v_n converges in BD to a harmonic function $v \in HBD$, and we have the decomposition

$$(u-\lambda)h=\varphi+v, \qquad \varphi \in \overline{K},$$

in addition

$$|v| \leq C(u-\lambda).$$

From this we see that $v(p_n)\to 0$, and it follows the next two cases:

$$v \equiv 0$$

or

$$v \not\equiv \text{constant}.$$

For the decomposition of A into $\overline{K}+HBD$ we can also verify that the unique element of $\overline{K} \cap HBD$ is the element which is the constant 0. Then the residue ring $A/\overline{K}$ is a normed ring, and moreover, if we define the multiplicative structure in HBD by means of the projection $f\to u$, where $f=\varphi+u$, $\varphi \in \overline{K}$, $u \in HBD$, then $A/\overline{K}$ is isomorphic to HBD. Taking this fact and the result obtained above into consideration, we can conclude that $u-\lambda$ has no inverse element in the residue ring $A/\overline{K}$. The necessary and sufficient condition that an element of a normed ring has the inverse element is that the element is not contained in any maximal ideal. On account of this condition, $u-\lambda$ is contained in a maximal ideal of the normed ring $A/\overline{K}$, which is equivalent to the fact that $u-\lambda$ belongs to a maximal ideal containing $\overline{K}$. This means that $u-\lambda=0$ on a point of Δ.

The proof for the maximum will be carried in the same way.

Remark. The theorem is true for the class HBD^* of these harmonic functions which are uniform limits of HBD functions.

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References

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