# 86. A Theorem on Modules of Trivial Cohomology over a Finite Group

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Let $G$ be a finite group. A (left, say) $G$-module $A$ is said to be of *trivial cohomology* when the cohomology groups $H^n(H, A)$ vanish for all $n \gtreqless 0$ and for all subgroups $H$ of $G$. It is known that a $G$-module $A$ is of trivial cohomology whenever there is an integer $r$ such that $H^r(H, A)=H^{r+1}(H, A)=0$ for all subgroups $H$ of $G$. This phenomenon was first noted in Hochschild-Nakayama [5] though for positive-dimensional cohomology groups only (the dimension "lowering" not explicitly given there works also by induction with respect to subgroups) and then independently by Lyndon (unpublished). The detailed proof of the machinery used, fundamental exact sequences, was given in Hochschild-Serre [6]. A simpler proof, making use of cohomological transfer maps, was given by Artin-Tate (Artin-Tate [1], Chevalley [3]). Its significance for Galois cohomology was observed first, by Hochschild-Nakayama [5], in its direct application, in connection of Tsen's theorem for instance, and then by Tate [8] in its application to the cohomology of class field theory, through Artin splitting modules.

Now, in the present note we wish first to note that in case of a $p$-group $G$ a $G$-module $A$ is of trivial cohomology as soon as $H^r(G, A)=H^{r+1}(G, A)=0$ for some integer $r$; observe that no assumption is made about the cohomology groups on proper subgroups. So, turning to the case of general finite groups, we see that the above theorem may be refined into:

**Theorem.** *Let $A$ be a $G$-module. If for every rational prime $p$ (dividing the order $[G]$ of $G$) there is an integer $r(p)$ such that $H^{r(p)}(H_p, A)=H^{r(p)+1}(H_p, A)=0$ for a Sylow subgroup $H_p$ of $G$ for $p$, then $A$ is of trivial cohomology, i.e.*

$$H^n(H, A)=0$$

*for all $n \gtreqless 0$ and for all subgroups $H$ of $G$.*

Leaving details and applications to a subsequent paper, we sketch our proof.

**Lemma 1.** Let $G$ be a $p$-group. Let $A$ be a $G$-module such that $H^r(G, A)=H^{r+1}(G, A)=0$ for some integer $r$. Then we have $H^r(G, A\otimes M)=H^{r+1}(G, A\otimes M)=0$ for any representation module $M$ of $G$ over the ring $Z$ of integers.

To prove this we first assume that $A$ is torsion-free. Let $N$ be a $G$-submodule of $M$ such that the residue-module $M/N$ is a minimal $G$-module. We have $qM \subset N$ for some prime $q$. In case $q \neq p$ we obtain readily $H^n(G, A\otimes M) \cong H^n(G, A\otimes N)$ for any $n$. On the other hand, if $q=p$ then $[M:N]=p$ and $M/N$ is $G$-isomorphic to $Z/pZ$, $Z$ being operated by $G$ trivially; for the 1-representation is the only irreducible representation of $G$ in a modular field of characteristic $p$ (see e.g. Brauer-Nesbitt [2]). We see readily $(A\otimes M)/(A\otimes N) \cong A/pA$ as $G$-modules; observe that $A$ is torsion-free. This gives an exact sequence $H^r(G, A\otimes N) \to H^r(G, A\otimes M) \to H^r(G, A/pA)$. Here the last term is 0, because of the exact sequence $H^r(G, A) \to H^r(G, A/pA) \to H^{r+1}(G, pA)$ whose extreme terms are 0; observe $pA \cong A$. It follows that $H^r(G, A\otimes M)$ is a homomorphic image of $H^r(G, A\otimes N)$.

Now, there is a representation module $M_0$ of $G$ over $Z$ such that the direct sum $M+M_0$ has a $G$-submodule $M_1$ which has the same $Z$-rank as $M+M_0$ and which is $G$-regular. Applying the above consideration to composition residue-modules of $(M+M_0)/M_1$, in place of $M/N$, we see that $H^r(G, A\otimes(M+M_0))$ is a homomorphic image of $H(G, A\otimes M_1)$. But the last group is 0, since $A\otimes M_1$ is $G$-regular together with $M_1$, and therefore $H^r(G, A\otimes(M+M_0))$ whence $H(G, A\otimes M)$ vanishes too.

In case $A$ is not torsion-free, we consider a free $G$-module $A_0$ of which $A$ is a homomorphic image. Denoting the kernel of the homomorphism by $A_1$, we have $H^{r+1}(G, A_1)=H^{r+2}(G, A_1)=0$. Applying the above to $A_1$, with $r$ replaced by $r+1$, we have $H^{r+1}(G, A_1\otimes M)=0$, which implies in turn $H^r(G, A\otimes M)=0$.

The assertion $H^{r+1}(G, A\otimes M)=0$ is proved similarly.

On taking as $M$ in Lemma 1 one of the dimension shifters $I$, $J$ of Chevalley [3] or their tensor product we have then

**Lemma 2.** Let $G$ be a $p$-group. Let $A$ be a $G$-module such that $H^r(G, A)=H^{r+1}(G, A)=0$ for some integer $r$. Then $H^n(G, A)=0$ for any $n \gtreqless 0$.

Having thus achieved the "dimension shifting" at the level of $G$ only, without referring to proper subgroups, we now turn to the "descent to subgroups". Let $G, A$ be as in Lemma 2. By the lemma we have $H^0(G, A)=H^1(G, A)=H^2(G, A)=0$ in particular. Let $H$ be a normal subgroup of $G$ such that $G/H$ is cyclic. The fundamental exact sequence (Artin-Tate [1])

$$0 \leftarrow H^0(G/H, A^H) \leftarrow H^0(G, A) \ (\leftarrow H^0(H, A))$$

implies then $H^0(G/H, A^H)=H^2(G/H, A^H)=0$. Then the fundamental exact sequence (Hochschild-Serre [6])

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^G \to H^2(G/H, A^H) \ (\to H^2(G, A))$$

implies $H^1(H, A)^G=0$. Then $H^1(H, A)$ itself must vanish, since the 1-representation is the only irreducible representation of $G$ in $Z/pZ$. Also $H^1(G/H, A^H)=H^3(G/H, A^H)=0$. Then the similar fundamental exact sequence with dimensions higher by one (which holds under the condition $H^1(H, A)=0$ (Hochschild-Serre [6])) implies $H^2(H, A)^G=0$ whence $H^2(H, A)=0$ too. By Lemma 2, with $H$ instead of $G$, we have $H^n(H, A)=0$ for all $n$.

By the repeated application of this argument we see that $A$ has the vanishing cohomology groups on any subgroup of $G$, which proves the $p$-group case of our theorem. The general case can be reduced to it by the Sylow group argument in cohomology (Artin-Tate [1]; Chevalley [3]; cf. also Hochschild-Nakayama [5], Lemma 1.5).

*Remark.* In the above proof we used $H^0(G, A)=H^1(G, A)=H^2(G, A)=0$ to obtain $H^1(H, A)=H^2(H, A)=0$. However, if we supplement the first of the fundamental exact sequences used above by its "transgression" part, to have

$$(*)\qquad 0\leftarrow H^0(G/H, A)\leftarrow H^0(G, A)\leftarrow H^0(H, A)_G\leftarrow H^{-1}(G/H, A^H)\quad(\leftarrow H^{-1}(G, A)),$$

and use this amplified sequence together with the second of the above fundamental exact sequences, then we can conclude $H^0(H, A)=H^1(H, A)=0$ from $H^0(G, A)=H^1(G, A)=0$ only. This would make a clear cut between the "dimension shifting" (by Lemma 2) and "descent to subgroups" (for dimensions 0 and 1, by means of the fundamental exact sequences). The validity of the exact sequence $(*)$ can be proved readily, and its significance as a predecessor of similar sequences for negative dimensions will be discussed in a subsequent note.

It might be of some interest to observe that the theorem of irreducible ($p$-)modular representations of $p$-groups was used in two ways in our proof, firstly in analyzing certain submodules of a tensor product and secondly in connection of the fundamental exact sequences. However, in the second application the theorem for a case of cyclic $p$-groups suffices, since $H^n(H, A)$ is operated essentially by the factor group $G/H$.

We want next to observe that that $G$ is a $p$-group is substantial in Lemma 2 (whence also in Lemma 1). Thus, *if $G$ is not a $p$-group there can in general exist a $G$-module $A$ which satisfies $H^r(G, A)=H^{r+1}(G, A)=0$ for a certain $r$ but for which $H^n(G, A)$ fails to vanish for some $n$.* For instance, let $G$ be a (finite) group ($\neq 1$) which coincides with its commutator subgroup $G'$. Then $H^{-2}(G, Z)(=H^2(G, Z))=G/G'=0$. Further $H^{-1}(G, Z)(=H(G, Z))=0$ as is well known. However $H^0(G, Z)=Z/[G]Z\neq 0$.

We want to observe also that *if $G$ is not a $p$-group then the* *cohomology groups $H^n(G, A)$ of $G$ in a $G$-module $A$ may, in general, well all vanish yet the cohomology groups $H^n(H, A)$ in $A$ on some subgroup $H$ of $G$ do not vanish.* Suppose, for instance, $G$ be the direct product $H\times L$ of two subgroups $H$, $L\neq 1$ such that a certain prime $p$ divides the order of $H$ while it does not divide the order of $L$. Let $A$ be the module of a (fully reducible) representation module of $L$ in the modular field $Z/pZ$ which does not contain the 1-representation as its constituent. We may naturally consider $A$ also as a $G$-module on which $H$ acts trivially. It follows readily that no irreducible constituent of the representation of $G$ defined by $A$ belongs to the 1-block (for $p$); Brauer-Nesbitt [2], § 29. By a theorem of Gaschütz [4] we have then

$$H^n(G, A)=0 \quad \text{for all } n\gtreqless 0;$$

though the theorem is proved there only for $n\geqq 1$, it, together with his proof to it, holds for all $n\gtreqless 0$. On the other hand, the $H$-module $A$ is simply isomorphic to a direct sum of a certain number of isomorphic copies of the $H$-module $Z/pZ$ (operated by $H$ trivially). Since $p$ divides $[H]$, we have $H^0(H, A)\neq 0$ for instance. Indeed, we have $H^n(H, A)\neq 0$ for all $n\gtreqless 0$ if $H$ is nilpotent, or more generally, if $H$ contains a normal subgroup $K$ such that $[H:K]$ is a power of $p$ while $[K]$ is prime to $p$. For, then $H^n(H, A)$ contains a subgroup isomorphic to $H^n(H/K, A^K)=H^n(H/K, A)$, by Hochschild-Serre [6], Theorem 1; again, though the theorem is proved there for $n\geqq 1$, it holds for $n\gtreqless 0$. But $H^n(H/K, Z/pZ)\neq 0$ for all $n\gtreqless 0$, by another theorem of Gaschütz [4] for instance. (We could apply the same theorem of Gaschütz directly to $H^n(H, A)$, to show $H^n(H, A)\neq 0$ for all $n$, on observing that the 1-representation is the only irreducible representation of $H$ belonging to the 1-block for $p$; see Brauer-Nesbitt [2], § 29.)

Finally, our Lemma 1 can easily be generalized to the case of any $[G]$-torsion free $G$-module $M$. When combined with the arguments of Tate [8], it leads to a proof of the writer's conjecture [7] concerning the cohomology of class field theory. This will be given also in a subsequent paper.

## References

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