164. *On the Cut Operation in Gentzen Calculi*

By Kiyoshi ISÉKI
Kobe University

(Comm. by K. KUNUGI, M.J.A., Dec. 13, 1956)

By *Gentzen's LK system,* we shall mean a formal system with the following schemata:

Axiom schema

$$\mathfrak{A} \to \mathfrak{A}$$

Logical rules of inferences

	in succedent	in antecedent
$\supset$	$\dfrac{\mathfrak{A}, \Gamma \to \theta, \mathfrak{B}}{\Gamma \to \theta, \mathfrak{A} \supset \mathfrak{B}}$	$\dfrac{\Gamma \to \theta, \mathfrak{A} \quad \mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \supset \mathfrak{B}, \Gamma \to \theta}$
&	$\dfrac{\Gamma \to \theta, \mathfrak{A} \quad \Gamma \to \theta, \mathfrak{B}}{\Gamma \to \theta, \mathfrak{A} \& \mathfrak{B}}$	$\dfrac{\mathfrak{A}, \Gamma \to \theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \to \theta} \quad \dfrac{\mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \to \theta}$
$\vee$	$\dfrac{\Gamma \to \theta, \mathfrak{A}}{\Gamma \to \theta, \mathfrak{A} \vee \mathfrak{B}} \quad \dfrac{\Gamma \to \theta, \mathfrak{B}}{\Gamma \to \theta, \mathfrak{A} \vee \mathfrak{B}}$	$\dfrac{\mathfrak{A}, \Gamma \to \theta \quad \mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \theta}$
$\neg$	$\dfrac{\mathfrak{A}, \Gamma \to \theta}{\Gamma \to \theta, \neg \mathfrak{A}}$	$\dfrac{\Gamma \to \theta, \mathfrak{A}}{\neg \mathfrak{A}, \Gamma \to \theta}$
$\forall$	$\dfrac{\Gamma \to \theta, \mathfrak{A}(a)}{\Gamma \to \theta, \forall x \mathfrak{A}(x)}$	$\dfrac{\mathfrak{A}(t), \Gamma \to \theta}{\forall x \mathfrak{A}(x), \Gamma \to \theta}$
$\exists$	$\dfrac{\Gamma \to \theta, \mathfrak{A}(t)}{\Gamma \to \theta, \exists x \mathfrak{A}(x)}$	$\dfrac{\mathfrak{A}(a), \Gamma \to \theta}{\exists x \mathfrak{A}(x), \Gamma \to \theta}$

Structural rules of inferences

	in succedent	in antecedent
Thinning	$\dfrac{\Gamma \to \theta}{\Gamma \to \theta, \mathfrak{A}}$	$\dfrac{\Gamma \to \theta}{\mathfrak{A}, \Gamma \to \theta}$
Contraction	$\dfrac{\Gamma \to \theta, \mathfrak{A}, \mathfrak{A}}{\Gamma \to \theta, \mathfrak{A}}$	$\dfrac{\mathfrak{A}, \mathfrak{A}, \Gamma \to \theta}{\mathfrak{A}, \Gamma \to \theta}$
Interchange	$\dfrac{\Gamma \to \Lambda, \mathfrak{A}, \mathfrak{B}, \theta}{\Gamma \to \Lambda, \mathfrak{B}, \mathfrak{A}, \theta}$	$\dfrac{\Gamma, \mathfrak{A}, \mathfrak{B}, \theta \to \Lambda}{\Gamma, \mathfrak{B}, \mathfrak{A}, \theta \to \Lambda}$

Cut

$$\frac{\Delta \to \Lambda, \mathfrak{A} \quad \mathfrak{A}, \Gamma \to \theta}{\Delta, \Gamma \to \Lambda, \theta}.$$

In the above schemata, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \cdots$ are arbitrary formulae, $a, b, c, \cdots$ are free variables and $s, t, \cdots$ are terms. $\Gamma, \Delta, \theta, \cdots$ are arbitrary finite sequences of zero or more formulae. *For the detail on the schemata,* see G. Gentzen [1], S. C. Kleene [2, 3].

In his papers, G. Gentzen [1] proved a principal theorem: any provable proposition in *LK*-system is provable without cut. And he observed that *the cut is equivalent to the mix rule*: let $\mathfrak{A}$ be a formula,

$\Pi, \Phi, \Sigma, \Omega$ are sequences of zero or more formulae such that Φ and Σ contain $\mathfrak{A}$.

$\Phi_{\mathfrak{A}}$ and $\Sigma_{\mathfrak{A}}$ are the results of dropping all $\mathfrak{A}$ in Φ and Σ respectively. Then

mix $$\frac{\Pi \to \Phi, \ \Sigma \to \Omega}{\Pi, \Sigma_{\mathfrak{A}} \to \Phi_{\mathfrak{A}}, \Omega}.$$

In this Note, we shall show the

Theorem 1. The cut in LK-system is replaced by

$$(*) \qquad \frac{\Gamma \to \mathfrak{A} \supset \mathfrak{B}, \Delta \quad \Pi, \Gamma \to \mathfrak{A}, \Xi}{\Gamma, \Pi \to \mathfrak{B}, \Delta, \Xi}.$$

Proof. Suppose the cut rule, then

$$\cfrac{\cfrac{\Gamma \to \Delta, \mathfrak{A} \supset \mathfrak{B} \quad \cfrac{\mathfrak{A} \to \mathfrak{A}, \ \mathfrak{B} \to \mathfrak{B}}{\mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \to \mathfrak{B}}}{\Gamma, \mathfrak{A} \to \Delta, \mathfrak{B}} \text{ cut}}{\cfrac{\Pi, \Gamma \to \mathfrak{A}, \Xi \quad \mathfrak{A}, \Gamma \to \Delta, \mathfrak{B}}{\Pi, \Gamma \to \mathfrak{B}, \Delta, \Xi} \text{ cut.}}$$

This shows $(*)$-rule. Conversely, assume that $\dfrac{\cdots\cdots}{\Delta \to \Lambda, \mathfrak{A} \quad \mathfrak{A}, \Gamma \to \theta}$, then

$$\cfrac{\cfrac{\cfrac{\mathfrak{A}, \Gamma \to \theta}{\Delta, \mathfrak{A}, \Gamma \to \theta}}{\Delta, \Gamma \to \mathfrak{A} \supset \theta \quad \Delta \to \Lambda, \mathfrak{A}}}{\Delta, \Gamma \to \Lambda, \theta}. \qquad (*)$$

Therefore, Theorem 1 is proved.

Theorem 2. Any provable proposition in LK-system is provable without $(*)$.

Theorem 2 is formulated in Gentzen's intuitionistic *LJ* system as follows.

Theorem 3. The cut of LJ-system is replaced by

$$(**) \qquad \frac{\Gamma \to \mathfrak{A} \supset \mathfrak{B} \quad \Gamma, \Pi \to \mathfrak{A}}{\Gamma, \Pi \to \mathfrak{B}}.$$

The inference rules of *LJ*-system are the following rules:

Axiom schema

$$\mathfrak{A} \to \mathfrak{A}$$

Logical rules of inferences

	in succedent	in antecedent
$\supset$	$\dfrac{\mathfrak{A}, \Gamma \to \mathfrak{B}}{\Gamma \to \mathfrak{A} \& \mathfrak{B}}$	$\dfrac{\Gamma \to \mathfrak{A}, \mathfrak{B}, \Pi \to \Delta}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Pi \to \Delta}$
$\&$	$\dfrac{\Gamma \to \mathfrak{A} \quad \Gamma \to \mathfrak{B}}{\Gamma \to \mathfrak{A} \& \mathfrak{B}}$	$\dfrac{\mathfrak{A}, \Gamma \to \theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \to \theta}$
		$\dfrac{\mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \to \theta}$

$\vee$ $$\frac{\Gamma \to \mathfrak{A}}{\Gamma \to \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\mathfrak{A}, \Gamma \to \theta \quad \mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \& \mathfrak{B},\ \Gamma \to \theta}$$

$$\frac{\Gamma \to \mathfrak{B}}{\Gamma \to \mathfrak{A} \vee \mathfrak{B}}$$

$\neg$ $$\frac{\mathfrak{A}, \Gamma \to}{\Gamma \to \neg\mathfrak{A}} \qquad \frac{\Gamma \to \mathfrak{A}}{\neg\mathfrak{A}, \Gamma \to}$$

$\forall$ $$\frac{\Gamma \to \mathfrak{A}(a)}{\Gamma \to \forall x \mathfrak{A}(x)} \qquad \frac{\mathfrak{A}(t), \Gamma \to \Theta}{\forall x \mathfrak{A}(x), \Gamma \to \theta}$$

$\exists$ $$\frac{\Gamma \to \mathfrak{A}(t)}{\Gamma \to \exists x \mathfrak{A}(x)} \qquad \frac{\mathfrak{A}(a), \Gamma \to \theta}{\exists x \mathfrak{A}(x), \Gamma \to \theta}$$

Structural rules of inferences

in succedent

$$\frac{\Gamma \to}{\Gamma \to \mathfrak{A}}$$

in antecedent

$$\frac{\Gamma \to \theta}{\mathfrak{A}, \Gamma}$$

$$\frac{\mathfrak{A}, \mathfrak{A}, \Gamma \to \theta}{\mathfrak{A}, \Gamma \to \theta}$$

$$\frac{\Gamma, \mathfrak{A}, \mathfrak{B}, \theta \to \Delta}{\Gamma, \mathfrak{B}, \mathfrak{A}, \theta \to \Delta}$$

Cut rule

$$\frac{\Gamma \to \mathfrak{A} \quad \mathfrak{A}, \Pi \to \theta}{\Gamma, \Pi \to \theta}.$$

The succedent for $\Gamma \to \Pi$ is empty or a formula. The restriction on variables in the schemata for $\forall$, $\exists$ is as usual. For the detail, see G. Gentzen [1].

The proof of Theorem 3 is very similar to Theorem 2. Thus we shall omit it.

References

[1] G. Gentzen: Untersuchungen über das logische Schliessen I, II, Math. Z., **39**, 176–210, 405–431 (1934–5).

[2] S. C. Kleene: Introduction to Metamathematics, New York (1952).

[3] S. C. Kleene: Permutability of inferences in Gentzen's calculi *LK* and *LJ*, Memoirs of Am. Math. Soc., No. 10 (1952).