3. *Complex Numbers with Vanishing Power Sums*

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1. By $\mathfrak{Z}_{m,n}$ we denote the set of systems of n complex numbers $(z_1, z_2, \cdots, z_n)$ with the property

$$s_\nu \equiv \sum_{j=1}^{n} z_j^\nu = 0 \qquad (\nu = m+1, m+2, \cdots, m+n-1)$$

for a prescribed non-negative integer m.

In a course of their study of the theory of Diophantine approximations Vera T. Sós and P. Turán*) were led to the problem of determining all the systems in $\mathfrak{Z}_{m,n}$, and proved that:

1° the systems in $\mathfrak{Z}_{0,n}$ are given by the zeros of an equation

$$z^n + a = 0 \qquad (a \text{ arbitrary complex});$$

2° the systems in $\mathfrak{Z}_{1,n}$ are given by the zeros of an equation

$$z^n + \frac{a}{1!} z^{n-1} + \cdots + \frac{a^n}{n!} = 0 \qquad (a \text{ arbitrary complex}); \text{ and}$$

3° the systems in $\mathfrak{Z}_{2,n}$ are formed by the zeros of an equation

$$z^n + \frac{H_1(\lambda)}{1!} a z^{n-1} + \cdots + \frac{H_n(\lambda)}{n!} a^n = 0,$$

where $H_\nu(t)$ stands for the νth Hermite polynomial defined by

$$H_\nu(t) = (-1)^\nu e^{t^2} \frac{d^\nu}{dt^\nu} e^{-t^2},$$

λ denotes any zero of the equation $H_{n+1}(t) = 0$ and a is an arbitrary complex number.

In the present note we wish to give a characterization of the systems in $\mathfrak{Z}_{m,n}$ for general integer values of $m > 0$.

2. We define polynomials $C_\nu = C_\nu(t_1, \cdots, t_m)$ $(\nu = 0, 1, 2, \cdots)$ by

$$\exp\left(-\sum_{\mu=1}^{m} \frac{1}{\mu} t_\mu x^\mu \right) = \sum_{\nu=0}^{\infty} \frac{C_\nu}{\nu!} x^\nu, \tag{1}$$

that is, by

$$C_\nu = \nu! \sum_{\substack{\mu_i \geq 0 \\ \mu_1 + 2\mu_2 + \cdots + m\mu_m = \nu}} \frac{\left(-\frac{t_1}{1}\right)^{\mu_1} \left(-\frac{t_2}{2}\right)^{\mu_2} \cdots \left(-\frac{t_m}{m}\right)^{\mu_m}}{\mu_1! \, \mu_2! \cdots \mu_m!}.$$

It is well known that the Hermite polynomials $H_\nu(t)$ $(\nu = 0, 1, 2, \cdots)$ are generated by

$$e^{2tx - x^2} = \sum_{\nu=0}^{\infty} \frac{H_\nu(t)}{\nu!} x^\nu.$$

*) Vera T. Sós and P. Turán: On some new theorems in the theory of Diophantine approximations, Acta Math. Acad. Sci. Hungar., **6**, 241–255 (1955).

Thus, for $m=2$ we have

$$C_\nu(-2u, 2v^2)=v^\nu H_\nu\left(\frac{u}{v}\right) \qquad (\nu=0,1,2,\cdots).$$

Now, our result can be stated as follows:

Theorem. *All the systems* $(z_1, z_2, \cdots, z_n)$ *in* $\mathfrak{Z}_{m,n}$ $(m>0)$ *are formed by the zeros of an equation*

$$\sum_{\nu=0}^{n} \frac{C_\nu(\lambda_1, \lambda_2, \cdots, \lambda_m)}{\nu!} z^{n-\nu}=0,$$

where $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ *is any solution of the system of equations*

$$C_\nu(t_1, t_2, \cdots, t_m)=0 \qquad (\nu=n+1, n+2, \cdots, n+m-1).$$

We note that the value of any one of the λ_i, λ_1 say, is arbitrarily given. Clearly our theorem covers the results 2° and 3° due to Sós and Turán.

3. Put

$$\prod_{j=1}^{n}(z-z_j)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n.$$

We are now going to determine the coefficients $a_1, \cdots, a_n$ under the condition

(2) $$s_{m+1}=s_{m+2}=\cdots=s_{m+n-1}=0.$$

There hold the recurrence formulae of Newton-Girard:

(3) $$s_\nu+s_{\nu-1}a_1+s_{\nu-2}a_2+\cdots+s_1a_{\nu-1}+\nu a_\nu=0$$

for $1\leqq\nu\leqq n$, and

(4) $$s_\nu+s_{\nu-1}a_1+\cdots+s_{\nu-n+1}a_{n-1}+s_{\nu-n}a_n=0$$

for $\nu>n$. It follows from this that, if $s_1, s_2, \cdots, s_n$ are given, then $a_1, a_2, \cdots, a_n$ are uniquely determined. Moreover, it is not difficult to see that

$$a_\nu=\sum_{\substack{\mu_i\geqq 0\\ \mu_1+2\mu_2+\cdots+n\mu_n=\nu}} \frac{\left(-\frac{s_1}{1}\right)^{\mu_1}\left(-\frac{s_2}{2}\right)^{\mu_2}\cdots\left(-\frac{s_n}{n}\right)^{\mu_n}}{\mu_1!\,\mu_2!\cdots\mu_n!} \qquad (1\leqq\nu\leqq n),$$

whence, putting $s_1=t_1$, $s_2=t_2, \cdots$, $s_m=t_m$ and using $s_{m+1}=s_{m+2}=\cdots=s_{m+n-1}=0$, we thus obtain

$$a_\nu=\frac{1}{\nu!}C_\nu(t_1, t_2, \cdots, t_m) \qquad (1\leqq\nu\leqq n).$$

Next, we shall show that these t_i's must satisfy the relations

$$C_{n+\kappa}(t_1, t_2, \cdots, t_m)=0 \qquad (\kappa=1,2,\cdots,m-1).$$

By differentiation with respect to x we get from (1)

(5) $$-\sum_{\mu=1}^{m} t_\mu x^{\mu-1}\sum_{\nu=0}^{\infty}\frac{C_\nu}{\nu!}x^\nu=\sum_{\nu=0}^{\infty}\frac{C_{\nu+1}}{\nu!}x^\nu.$$

Put $m_\kappa=\min(m, n+\kappa)$ for $1\leqq\kappa\leqq m-1$. The comparison of the coefficients of x^n on both sides of (5) gives

$$\frac{C_{n+1}}{n!}=-\left(t_1\frac{C_n}{n!}+\cdots+t_{m_1}\frac{C_{n+m_1-1}}{(n+m_1-1)!}\right)$$

$$=-(t_1a_n+\cdots+t_{m_1}a_{n+m_1-1}+\cdots+s_na_1+s_{n+1})$$
$$=0,$$

by (2) and (4). Thus $C_{n+1}=0$. Now suppose that $C_{n+1}=\cdots=C_{n+\kappa-1}=0$. Again, by the comparison of the coefficients of $x^{n+\kappa-1}$ on both sides of (5) and using (4) we find that

$$\frac{C_{n+\kappa}}{(n+\kappa-1)!}=-\left(t_1\frac{C_{n+\kappa-1}}{(n+\kappa-1)!}+\cdots+t_{m_\kappa}\frac{C_{n+m_\kappa-\kappa}}{(n+m_\kappa-\kappa)!}\right)$$
$$=-(t_\kappa a_n+\cdots+t_{m_\kappa}a_{n+m_\kappa-\kappa}+\cdots+s_{n+\kappa-1}a_1+s_{n+\kappa})$$
$$=0,$$

whence $C_{n+\kappa}=0$, and our assertion is proved by induction.

Conversely, let $z_1, z_2,\cdots, z_n$ be the zeros of an equation of the type described in the theorem. Then, by a similar argument as above, we can show that the system $(z_1, z_2,\cdots, z_n)$ satisfies the relation (2), using (3), (4) and (5). This concludes the proof of our theorem.