25. On Pseudo-compact and Countably Compact Spaces

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(Comm. by K. KUNUGI, M.J.A., Feb. 12, 1957)

In his kind letter of January 13, 1957 to S. Kasahara, one of the present writers, Prof. S. Mardešić of the University of Zagreb, Yugoslavia, communicated an interesting characterisation of pseudo-compact without proof by S. Mrówka. The result stated which is due to him is the following

*Theorem. A completely regular space is pseudo-compact if and only if every locally finite open covering has a finite subcovering.**)

The concept of pseudo-compact space was introduced by E. Hewitt [2]. A completely regular space is said to be *pseudo-compact*, if every real continuous function on it is bounded.

In this Note, we shall first give a simple proof of Theorem. To prove it, we shall prove the following

Theorem 1. The following properties of a completely regular space S are equivalent:

(1) *S is pseudo-compact.*

(2) *Every locally finite open covering has a finite subcovering.*

(3) *Every star finite open covering has a finite subcovering.*

Proof. To prove the implication (1)→(2), let $\sigma=\{O_\alpha\}$ be a locally finite open covering of S. Suppose that σ has no finite subcovering, then we can find a denumerable subfamily $\{O_n\}$ of σ which every finite family of it does not cover S. With each O_n, we associate a certain point $a_n\in O_n$. Since S is completely regular, for every n, we can find a non-negative continuous function $f_n(x)$ such that $f_n(a_n)=n$ and $f_n(x)=0$ for $x\in S-O_n$. Since σ is locally finite $f(x)=\sum_{n=1}^{\infty}f_n(x)$ is well-defined and continuous on S. On the other hand, $f(a_n)\geq n$, and hence $f(x)$ is unbounded continuous, which is a contradiction to the hypothesis. Therefore we have (1)→(2).

The implication (2)→(3) is trivial, since every star finite open covering is locally finite.

To prove (3)→(1), we shall show that any non-negative continuous function $f(x)$ is bounded. It is obvious that it leads the pseudo-compactness of S. By the continuity of $f(x)$, the sets $O_1=\{x\mid f(x)<2\}$, $O_n=\{x\mid n-1<f(x)<n+1\}$ $(n=2,3,\cdots)$ are open. The family $\{O_n\}$ is

*) For various terminologies, see J. L. Kelley: General Topology, New York (1955).

an open covering of S, and each O_n does not meet O_i ($i \neq n-1, n, n+1$). Hence the covering $\{O_n\}$ is star finite. Therefore $\{O_n\}$ has a finite subcovering $\{O_{n_i}\}$ ($i=1, 2, \cdots, m$). Hence we have $f(x) < \mathrm{Max}(n_1, n_2, \cdots, n_m)$, and $f(x)$ is bounded. We have a proof of Theorem 1.

Remark. The given coverings in Theorem 1 may be replaced by countable many. The proofs are very similar with it. Therefore, for example, we have

(1) S is pseudo-compact;

(2) every locally finite countable open covering has a finite subcovering;

(3) every star finite countable open covering has a finite subcovering. These conditions above are equivalent for a completely regular space S.

Next, we shall consider the case that every point finite open covering has a finite subcovering. Then we have the following

Theorem 2. The following three properties of a regular T_1-space S are equivalent:

(1) *Every point finite open covering of S has a finite subcovering.*

(2) *Every point finite countable open covering of S has a finite subcovering.*

(3) *S is countably compact.*

Proof. It is sufficient to show that (2) implies (3), and (3) implies (1), since the implication (1)$\rightarrow$(2) is trivial.

To prove that (2) implies (3), let us suppose that the space is not countably compact. Then there is a sequence $\{x_n\}$ of points of S such that $\{x_n\}$ has no cluster point. Since x_1 is not a cluster point of $\{x_n\}$, we can find a closed neighbourhood V_1 of x_1 which does not contain $x_2, x_3, \cdots, x_n, \cdots$ by the regularity of S. Suppose that we could construct pairwise disjoint closed neighbourhoods V_i of x_i ($i=1, 2, \cdots, n$) not containing x_k ($k>n$), then $V_1 \cup V_2 \cup \cdots \cup V_n$ being closed, there is a closed neighbourhood V_{n+1} of x_{n+1} such that

$$V_{n+1} \not\ni x_{n+j} \quad (j=2, 3, \cdots)$$

and

$$V_{n+1} \subset S-(V_1 \cup V_2 \cup \cdots \cup V_n).$$

Thus, to each point x_i we can assign a closed neighbourhood V_i such that the neighbourhoods $\{V_i\}$ are pairwise disjoint and $V_i \not\ni x_j$ for $i \neq j$. As can be easily seen, the complement of $\{x_n\}$ and the interiors of V_i make a point finite countable open covering of S which has no finite subcovering. This leads to a contradiction.

As to the implication (3)$\rightarrow$(1), it is implicitly contained in the proof of Theorem 2.4 of a paper by R. Arens and J. Dugundji [1]. Therefore we shall omit the detail of it.

E. Hewitt [2] has proved that *a normal space is pseudo-compact* *if and only if it is countably compact.* Thus, by Theorem 1 and Theorem 2, we have the following

Theorem 3. The following statements for a normal space S are equivalent:

(1) *S is pseudo-compact.*

(2) *S is countably compact.*

(3) *Every star finite* (*countable*) *open covering has a finite subcovering.*

(4) *Every locally finite* (*countable*) *open covering has a finite subcovering.*

(5) *Every point finite* (*countable*) *open covering has a finite subcovering.*

References

[1] R. Arens and J. Dugundji: Remark on the concept of compactness, Portugaliae Math., **9**, 141–143 (1950).

[2] E. Hewitt: Rings of real-valued continuous functions. I, Trans. Am. Math. Soc., **64**, 45–99 (1948).