24. *On the Cut Operation in Gentzen Calculi. II*

By Kiyoshi ISÉKI

(Comm. by K. KUNUGI, M.J.A., Feb. 12, 1957)

The object of this Note is to give an exact form of Theorem 1 in my Note [1]. Theorem 1 is incorrect and its proof is not sufficient. We follow the terminologies and notations in my Note [1] in the sequel.

Theorem 1. The cut rule in LK-system is replaced by

(1) $$\frac{\Gamma\to\mathfrak{A}\supset\mathfrak{B},\Delta \quad \Pi,\Gamma\to\mathfrak{A}}{\Gamma,\Pi\to\Delta,\mathfrak{B}}$$

and

(2) $$\frac{\to\mathfrak{A} \quad \mathfrak{A}\to}{\to}$$

Proof. In my Note [1], we proved that the cut rule implies (1), and (2) follows from the cut rule immediately. To prove that (1) and (2) imply the cut rule:

(3) $$\frac{\Gamma\to\Delta,\mathfrak{A} \quad \mathfrak{A},\Pi\to\Lambda}{\Gamma,\Pi\to\Delta,\Lambda}$$

If Λ is not empty, there is a proposition $\mathfrak{B}$ in Λ. Then we have the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A},\Pi\to\Lambda}{\mathfrak{A},\Pi\to\Lambda_{\mathfrak{B}},\mathfrak{B}}}{\Pi\to\Lambda_{\mathfrak{B}},\mathfrak{A}\supset\mathfrak{B}} \quad \Gamma\to\mathfrak{A},\Delta}{\Pi,\Gamma\to\Lambda_{\mathfrak{B}},\Delta,\mathfrak{B}}}{\Pi,\Gamma\to\Delta,\Lambda}$$

In (3), if Λ is empty, we have

(4) $$\frac{\Gamma\to\Delta,\mathfrak{A} \quad \mathfrak{A},\Pi\to}{\Gamma,\Pi\to\Delta}$$

If Π is not empty, Π contains a proposition $\mathfrak{B}$, and then we have,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A},\Pi\to}{\mathfrak{A},\mathfrak{B},\Pi_{\mathfrak{B}}\to}}{\mathfrak{A},\Pi_{\mathfrak{B}}\to\neg\mathfrak{B}}}{\Pi_{\mathfrak{B}}\to\mathfrak{A}\supset\neg\mathfrak{B}} \quad \Gamma\to\Delta,\mathfrak{A}}{\Gamma,\Pi_{\mathfrak{B}}\to\Delta,\neg\mathfrak{B}}}{\Gamma,\Pi_{\mathfrak{B}},\mathfrak{B}\to\Delta}}{\Gamma,\Pi\to\Delta}$$

This shows (4). If Π is empty, cut rule is

$$\frac{\Gamma\to\Delta,\mathfrak{A} \quad \mathfrak{A}\to}{\Gamma,\to\Delta}$$

If $\varDelta$ is not empty, the proof of the first case is applicable,

$$\dfrac{\dfrac{\dfrac{\mathfrak{A}\to}{\Gamma\to\varDelta,\mathfrak{A}}\quad \mathfrak{A}\to\varDelta}{\Gamma\to\varDelta,\varDelta}}{\Gamma\to\varDelta}$$

This proof is also available for the second case. Finally, we suppose that $\varDelta$ is empty and Γ is not empty. Therefore the cut rule is

$$\dfrac{\Gamma\to\mathfrak{A}\quad \mathfrak{A}\to}{\Gamma\to}$$

if $\mathfrak{B}$ is contained in Γ, then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A}\to}{\mathfrak{A}\to\neg\mathfrak{B}}}{\to\mathfrak{A}\supset\neg\mathfrak{B}}\quad \Gamma\to\mathfrak{A}}{\Gamma\to\neg\mathfrak{B}}}{\Gamma,\mathfrak{B}\to}}{\Gamma\to}$$

Therefore Theorem 1 is obtained with (2).
Similarly, we have the following

Theorem 2. The cut rule in LJ-system is replaced by

(5) $$\dfrac{\Gamma\to\mathfrak{A}\supset\mathfrak{B}\quad \Gamma,\Pi\to\mathfrak{A}}{\Gamma,\Pi\to\mathfrak{B}}$$

(6) $$\dfrac{\to\mathfrak{A}\quad \mathfrak{A}\to}{\to}$$

Therefore we have

Theorem 3. Any provable proposition in LK-system is provable without (1) and (2). In LJ-system, without (5) and (6).

Reference

[1] K. Iséki: On the cut operation in Gentzen calculi, Proc. Japan Acad., **32**, 719–721 (1956).