22. A Note on the Singular Homotopy Type of Spaces

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Our purpose of this paper is to generalize a result of H. Suzuki [4] concerning the homotopy type of a space and its loop space.

I. Let M be a c.s.s. complex (complete semi-simplicial complex), Π be an abelian group, and $n\geq 2$ be an integer. Let $\mathfrak{K}^{n+1}\in H^{n+1}(M,\Pi)$ be a cohomology class, and $X=K(M,\Pi;\mathfrak{K}^{n+1})$ be the c.s.s. complex defined in the paper [2], §1. Let G be an abelian group. Denote by A the normalized cochain group $C_N^*(X,G)$ of X. As was shown in the paper [2], §5, there is a filtration $A=\smile A^r$, such that the term $E_2^{p,q}$ of the spectral sequence $\{E_r\}$ derived from this filtration is canonically isomorphic with $H^p(M,H^q(\Pi,n;G))$. In the sequel, we assume that M operates trivially on Π and G. Let $p_r^*:H^r(M,G)\to H^r(A^1,G)$ be the homomorphism induced by the projection $p:X\to M$ and let $\delta_{r-1}:H^{r-1}(\Pi,n;G)\to H^r(A^1,G)$ be the coboundary homomorphism. Then, the transgression $t_{r-1}^*:\delta_{r-1}^{-1}(\text{image } p_r^*)\to H^r(M,G)/(\text{kernel } p_r^*)$ is defined by $t_{r-1}^*=p_r^{*-1}\delta_{r-1}$. Especially, t_n^* is a homomorphism of $H^n(\Pi,n;G)$ into $H^{n+1}(M,G)$ [3].

Lemma 1. Let I be the identity automorphism $\in Hom(\Pi,\Pi)$, and $t_n^*:Hom(\Pi,\Pi)\approx H^n(\Pi,n;\Pi)\to H^{n+1}(M,\Pi)$ be the transgression. Then,

$$\mathfrak{K}^{n+1}=t_n^*(I).$$

II. Let X be a simply connected space, E the space of paths in X starting at a fixed point $x_0\in X$, and Ω be the loop space $\subseteq E$. Let $p_r^*:H^r(X,G)\to H^r(E,\Omega;G)$ be induced by the projection $p:E\to X$ and $\delta_r:H^r(\Omega,G)\to H^{r+1}(E,\Omega;G)$ be the coboundary homomorphism. Then, the suspension $S_r:H^{r+1}(X,G)\to H^r(\Omega,G)$ is defined by $S_r=\delta_r^{-1}p_r^*$. If X is p-connected, S_r is an isomorphism (into or onto) for $0<r<2\times p$.

Let X be a simply connected space, $X_{(n)}$ be the n-combined space of X (§1, [1]). Since $X_{(n)}$ is obtained by attaching cells to X, we may assume that

$$X\subseteq\cdots\subseteq X_{(n+1)}\subseteq X_{(n)}\subseteq\cdots\ .$$

Furthermore, we may assume that $X_{(n+1)}$ is a fibre space over $X_{(n)}$.[1] The minimal complex $M_{(n+1)}$ of $X_{(n+1)}$ is simplicial isomorphic to the c.s.s. complex $K(M_{(n)},\pi_{n+1}(X);\mathfrak{K}^{n+2}(X))$ (§2, [1]).

The loop space $\Omega(X_{(n)})$ of $X_{(n)}$ is the $(n-1)$-combined space of

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1) There is a fibre space E over $X_{(n)}$ such that E has the same homotopy type with $X_{(n+1)}$ and the projection $p:E\to X_{(n)}$ is equivalent to the inclusion $X_{(n+1)}\subseteq X_{(n)}$ [1].

the loop space $\Omega(X)$ of X.

Lemma 2. In the diagram:

$$\begin{array}{ccc} H^n(\pi_n(\Omega(X_{(n)})), n; G) & \xrightarrow{\bar{t}^*} & H^{n+1}(\Omega(X_{(n)}); G) \\ \uparrow \bar{S} & & \uparrow S \\ H^{n+1}(\pi_{n+1}(X), n+1; G) & \xrightarrow{t^*} & H^{n+2}(X_{(n)}; G), \end{array}$$

the relation

$$t^*\bar{S} = -St^*$$

holds, where $S, \bar{S}$[2] *are the suspensions, and* $t^*, \bar{t}^*$ *are the transgressions.*

III. Let X be simply connected, $\Omega(X)$ be the loop space of X and $\mathfrak{K}^n, \bar{\mathfrak{K}}^n$ be the n-generalized Eilenberg-MacLane invariants of X and $\Omega(X)$, respectively (§ 3, [2]). Let $S_{n+1}: H^{n+2}(X_{(n)}, \pi_{n+1}(X)) \to H^{n+1}(\Omega(X_{(n)}), \pi_{n+1}(X))$ be the suspension. Then, by Lemmas 1 and 2, we have

$$S_{n+1}\mathfrak{K}^{n+2} = -\bar{\mathfrak{K}}^{n+1}$$

By this relation, the following theorem readily follows from Theorem 4. 2.

Theorem. Let X and Y be two p-connected spaces ($p \geq 1$) and let q be an integer such that $p \leq q \leq 2p-1$. If $\Omega(X)$ and $\Omega(Y)$ have the same singular q-homotopy type, then X and Y have the same singular $(q+1)$-homotopy type.

Corollary. Let X and Y be two A_n^{n-1} polyhedra ($n \geq 1$). If $\Omega(X)$ and $\Omega(Y)$ have the same singular $(2n-2)$-homotopy type, then X and Y have the same homotopy type.

References

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[2] Y. Inoue: A complete system of invariants of singular homotopy type (to appear).

[3] J. P. Serre: Homologie singulière des espaces fibrés. Applications, Ann. Math., **54**, 425-505 (1951).

[4] H. Suzuki: On the Eilenberg-MacLane invariants of loop spaces, Jour. Math. Soc. Japan, **8**, 93-101 (1956).

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2) Let $p: X_{(n+1)} \to X_{(n)}$ be the projection, and $F = p^{-1}(x_0)$ be a fibre. Then, $\bar{S}$ is the suspension in the fibre space (E_F, q, F), where E_F is a space of paths in F starting at a fixed point $\in F$.